Efficiency Through Revenue Redistribution in Auctions with Entry*

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Abstract

We consider a single-item private values auction setting with endogenous costly entry. We demonstrate that for symmetric bidders and entry costs that are an arbitrary linear function of value, a unique symmetric Bayes-Nash "cutoff equilibrium" (where only agents with value above some threshold enter) obtains under the Vickrey auction and also a broader class of revenue-redistributing auctions. We show that when there are participation externalities and entry costs decrease as participation increases (e.g., when there is a search cost associated with obscure auctions), the Vickrey auction is inefficient and a revenue-redistributing auction dominates. We characterize welfare-maximizing allocatively-efficient auctions in such settings, and identify conditions under which the optimum can be achieved without running a deficit.

1 Introduction

It is common in the study of auctions to assume a pre-determined set of participants. A richer model, by now also the subject of significant attention, considers the set of participants to be a variable that itself is dependent on the nature of the auction. Making the model rich in this way is especially crucial for settings where agents incur a cost for participation ("entry costs"), since rational agents will not participate in an auction that does not produce expected payoff exceeding entry cost.[1]

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[1]For instance, if it costs an agent \$10 to drive to an auction and \$5 for the time spent at the auction, he must expect to gain at least \$15 from the auction or else participation is not rational.

There are several important facts known about auction design with entry costs. McAfee and McMillan [12], adopting a model where values are learned only *after* deciding to participate and costs are constant across agents and known in advance, show that a first-price sealed bid auction elicits the number of bidders that is optimal with respect to expected revenue. Moreno and Wooders [14] address revenue maximization in a variant with heterogenous entry costs; other examples of work in this model include Tan [18] and Levin and Smith [10].

The model considered in the current paper, as well as in the references that follow, instead assumes each agent knows his value and entry cost prior to making a participation decision. The second-price sealed-bid (Vickrey) auction receives particular attention in the literature, and with good reason. Green and Laffont [6] consider entry costs that can vary arbitrarily across agents, and show that in the case where values and costs are both distributed uniformly, an equilibrium exists in which agents participate if and only if cost is less than some function of value. Stegeman [17] shows that for arbitrary individualized costs for the various agents, the Vickrey auction has an equilibrium that is efficient in the sense of eliciting participation that maximizes expected allocation value minus participation costs.[2]

It is common in much of the literature to assume valuations drawn from the same distribution (symmetric bidders) and participation costs that are *constant* across bidders. In a very early precursor to Stegeman's result, Samuelson [16] shows that in such settings the Vickrey auction (as well as other sealed-bid auctions) with a reserve price equal to seller value has a symmetric equilibrium that maximizes social welfare. Tan and Yilankaya [19]—also in a setting with constant costs and a common value distribution—show that the Vickrey auction has a *unique* equilibrium if the cumulative distribution function over values is concave. Even without this concavity assumption, there is always exactly one *symmetric* equilibrium, so their result amounts to a demonstration that with concavity there are no asymmetric equilibria.[3]

In general the efficient equilibrium of the Vickrey auction with entry costs may be asymmetric, but the combination of the Stegeman and the Tan and Yilankaya results implies that with a concave symmetric value distribution and a fixed entry cost the Vickrey auction has a unique equilibrium, and it is both symmetric and efficient. As in [6], equilibria in these settings have the form of a "cutoff value", where an agent participates if and only if his value is above some critical level v_0; so in a symmetric cutoff equilibrium all agents with value above some v_0 will participate, and all others will not.

Symmetric equilibria are intuitively more compelling than asymmetric ones when there is a common distribution over values, since there is no a priori reason to distinguish one agent's expected behavior from another's. However, the assumption of constant costs is uncomfortably restrictive since it cannot model settings where, say, it is more costly to prepare a bid given a higher value, or where higher-valued agents bear more opportunity cost in terms of time spent at the auction. Thus in

[2]Stegeman also shows that the first-price sealed-bid auction may have no efficient equilibrium.

[3]Menezes and Monteiro [13] also consider the constant cost, common value distribution setting, analyzing equilibria of first and second-price auctions and designing a revenue-optimal auction.

this paper we adopt a model that allows us to generalize some of the core results that assumed constant entry costs while maintaining enough structure to preserve a (unique) symmetric cutoff equilibrium; specifically we will consider costs that are linearly dependent on value.[4] While Stegeman's result [17] is general enough to tell us that the Vickrey auction has an efficient equilibrium here, we will show that moreover the *symmetric* equilibrium it achieves is superior to all other symmetric cutoff values (regardless of whether there are asymmetric equilibria).

But the main focus of the paper is to consider cases in which alternatives to the Vickrey auction, despite Stegeman's striking result, may be called for. Specifically, we consider settings where *increasing* participation over the Vickrey auction level is desirable. There are many scenarios that could generate this goal. For instance, consider the case where the agents' perception of cost is different than that of the social planner (think of "lazy" agents, or those whose behavior is unduly subject to inertia). Here, additional participation incentives are required to restore efficiency. Perhaps even more compelling is the case where agents and the social planner evaluate costs the same way, but costs are dependent on the participation level—i.e., there are *positive externalities* to participation. We will consider "obscurity costs", where an obscure (relatively sparsely attended) auction imposes greater costs on participants than one in which participation is widespread. We will show that the Vickrey auction is inefficient here due to underparticipation.

In looking for more efficient alternatives to the Vickrey auction for these settings, it is natural to restrict our focus to auction formats that—in equilibrium—allocate the good to the agent with the highest value, and are thus Pareto-efficient given the constraints imposed by the participation decisions (we will call such auctions *allocatively-efficient*). By the Revenue-Equivalence Theorem [20; 15], all allocatively-efficient auctions differ only by a constant payment provided to each bidder; therefore it is sufficient to consider only modifications of the Vickrey auction in which each participant receives a "bonus" payment that is independent of his bid. But it is important to define the participation payments in a way that does not lead to a deficit, as otherwise the auction would be infeasible in the absence of an external budget. Thus simply applying *fixed* participation bonuses will not be a solution because they may exceed the revenue generated by the Vickrey auction.

For the case of single-item allocation a non-deficit Vickrey auction alternative is presented by Bailey [1] (and also Cavallo [3]) in the form of a *redistribution auction* (*RA*) that takes the revenue of the Vickrey auction and returns a large portion of it to the agents without distorting incentives or running a deficit.[5] We will expand the *RA* auction conception to define a parameterized spectrum of auctions (RA_ρ, for $\rho \in [0, 1]$) that includes the Vickrey auction at one end ($\rho = 0$), the RA auction at the other ($\rho = 1$), and gradations of revenue redistribution in between; we will also

[4] [9] and [2], though not addressing an auction context, are other examples where participation cost is taken to be a function of type.

[5] Cavallo [3] provides a general *redistribution mechanism* that is truthful and efficient in dominant strategies, ex post individually rational, and no-deficit for all non-negative value typespaces. The mechanism coincides with that of Bailey [1] in the case of single-item allocation if each agent's typespace admits the value 0 for the item, but is also applicable to important settings where Bailey's mechanism is not, such as combinatorial allocation.

expand it to allow specification of a reserve price, and with it enlarged redistribution payments. We will do equilibrium analysis and compare expectations of efficiency under the Vickrey auction and the redistribution auction variants.

Our first contribution is to extend previous equilibrium results to a non-fixed cost setting: we show that for arbitrary cost functions that are linearly dependent on value, the Vickrey auction as well as a set of redistribution auctions each have a unique symmetric cutoff equilibrium (Theorem 2); notably, redistribution auctions achieve greater participation (and thus higher expected allocation value) than the Vickrey auction. Moreover, we show that under a mild condition on the value distribution, adding a positive reserve price to the redistribution auction yields yet *greater* participation and expected allocation value (Theorem 4), while remaining non-deficit in equilibrium.

We show that the Vickrey auction yields the socially optimal symmetric cutoff strategy in equilibrium for arbitrary cost functions that are linear in agent valuations (Theorem 5), but this does not hold for cost functions with participation externalities. This brings us to the main emphasis of the paper: analysis of cost functions that are strictly *decreasing* in the expected number of participating agents. For such settings we identify redistribution auctions that yield strictly greater efficiency than the Vickrey auction in a unique symmetric equilibrium without running a deficit (Theorem 8); we also characterize the *welfare-optimal* allocatively-efficient auction (Theorem 9), which may or may not be implementable without running a deficit, depending on the cost function (Theorems 11 and 12).

The paper proceeds as follows: in Section 2 we formally define the setting, the notation to be used, and the auction mechanisms to be considered. In Section 3 we present results for linear cost functions. Section 4 contains the majority of the paper's major results, which address the positive participation externality model where cost is a decreasing function of participation rate. In Section 5 we look beyond cost functions that are linearly dependent on value, and in Section 6 we conclude. Several of the proofs are deferred to the Appendix in the interest of readability.

2 Preliminaries

A single item is to be allocated and there is a set $I = \{1, \ldots, N\}$ of risk-neutral agents with independent private values for it; the seller is assumed to have no value for the good.[6] We will consider the possibility that only a subset of agents participate in the allocation mechanism, and we will use $n \leq N$ to denote the number of bidders (participants). We will focus on the following two auction forms, where the second is an auction schema and reduces to the first if $\rho = 0$.

Definition 1. (Vickrey auction with reserve price r.) *Each agent communicates a bid. If the highest bid exceeds r, the item is allocated to the highest bidder, who*

[6] Consider, e.g., government allocation of a public resource that can only be exploited by the agents.

pays the maximum of r and the second highest bid. No other agents make or receive payments.

Definition 2. (redistribution auction RA_ρ, with parameter $\rho \in [0, 1]$ and reserve price r). *Each agent communicates a bid. If the highest bid exceeds r, the item is allocated to the highest bidder, who pays the maximum of r and the second highest bid. Each agent i (including the high bidder) is then payed $\frac{\rho}{n}$ times the maximum of r and the second highest bid amongst agents other than i.*

RA_ρ (for $\rho > 0$) modifies the Vickrey auction by returning revenue to each agent in a way that does not depend on the agent's bid, and thus does not alter the agent's bidding incentives. Since the Vickrey auction, with or without a reserve price, yields truthful bidding as a dominant strategy, so too does RA_ρ:

Proposition 1. *For any $\rho \in \Re$, for any $r \in \Re$, RA_ρ with reserve price r yields truthful bidding as a dominant strategy for each agent.*

When the reserve price is set to 0, since RA_ρ (for any ρ) allocates the good to the highest bidder, it is therefore efficient in dominant strategies. We will be doing equilibrium analysis of the social welfare properties of these auctions and Proposition 1 will greatly simplify our analysis since, though there are two "stages" to agent strategies (choice to participate or not, and then bid choice), contingent on participation truthfulness is always a dominant strategy in the second stage. The following proposition, which follows almost immediately from Definition 2, entails that a deficit will never result in equilibrium under an RA_ρ auction if only agents with value exceeding the reserve price are incentivized to bid:[7]

Proposition 2. $\forall \rho \in [0, 1]$, *$\forall r \in \Re$, RA_ρ with reserve price r does not run a deficit if all bids meet or exceed r.*

We now formally describe the properties of our setting. Each of the N agents is aware of the form of the auction and the number of agents N. Each agent has an individual entry cost determined by his value; cost is defined by function $c : [0, 1] \to (0, 1)$ (the same for all agents) mapping value to cost, with $c(v)$ denoting the cost that an agent with value v bears for participation. Each agent privately observes his value (and thus his cost) and decides whether or not to participate. Utilities are quasilinear, so letting x_i be 1 if agent i with value v_i is allocated the item (at price p) and 0 otherwise, a participating agent i's utility equals $x_i(v_i - p) - c(v_i)$; a non-participating agent's utility is 0.

Values are independently and identically distributed over the interval $[0, 1]$ according to density function f, with F denoting the cumulative distribution function. We define G to be the c.d.f. for the highest value amongst $N - 1$ agents (i.e, $G(x) = F(x)^{N-1}$) and g the p.d.f. We define H_m^z and h_m^z to be the c.d.f. and p.d.f., respectively, for the z^{th} highest value amongst m agents, with $H_m^z(\cdot, v)$ and $h_m^z(\cdot, v)$

[7]Proof sketch: If the high bid is at least r, each of the n bidders receives a redistribution payment no greater than $1/n$ times the Vickrey auction revenue, and thus total redistribution never exceeds Vickrey auction revenue.

denoting the c.d.f. and p.d.f. conditioned on the m values all being at least v. We let $\mathcal{P}(N, n, v)$ denote the probability that exactly n out of N agents have value at least v, i.e.,

$$\mathcal{P}(N, n, v) = \binom{N}{n} (1 - F(v))^n F(v)^{N-n} \tag{1}$$

Finally, since we will be analyzing redistribution auctions, we define notation $R_\rho(v_0, r)$ to be the expected redistribution payment to each agent under RA_ρ with reserve price r, under the condition that every agent participates if and only if his value is at least v_0. In words, $R_\rho(v_0, r)$ is the expected value of: the second highest value at least as great as v_0 amongst $N - 1$ agents (or r if there is none) divided by one plus the number of those agents with value at least v_0. That is, for a zero reserve price,

$$R_\rho(v_0, 0) = \sum_{m=2}^{N-1} \mathcal{P}(N-1, m, v_0) \frac{\rho}{m+1} \int_{v_0}^{1} h_m^2(x, v_0) x \, dx \tag{2}$$

And, more generally, for arbitrary $r \geq 0$,

$$R_\rho(v_0, r) = \mathcal{P}(N-1, 0, v_0)\, \rho r \; + \; \mathcal{P}(N-1, 1, v_0)\, \frac{\rho}{2} r + \tag{3}$$

$$\sum_{m=2}^{N-1} \mathcal{P}(N-1, m, v_0) \frac{\rho}{m+1} \int_{v_0}^{1} h_m^2(x, v_0) x \, dx \tag{4}$$

Since expected social welfare depends on expected participation, and since expected participation depends on expected payoff for an agent, which in turn depends on the strategic participation decisions of other agents, we will form expectations based on what occurs in a game theoretic equilibrium. Specifically, we will be interested in outcomes that occur in Bayes-Nash equilibrium, where, given knowledge of the value density function f, no agent can expect to gain from deviating from the posited strategy given that other agents do not. We will find that equilibria have the form of a *cutoff level*, where agents participate if and only if they have values above the cutoff. We will use the term *Bayes-Nash cutoff equilibrium* to refer to a Bayes-Nash equilibrium that has this property. Formally:

Definition 3 (Bayes-Nash cutoff equilibrium)**.** *Given a common-knowledge distribution over each agent's valuation and a vector of private values $v_1, \ldots, v_n$, bidding strategies $\sigma_1, \ldots, \sigma_n$, with $\sigma_i : \Re \to \Re$, $\forall i \in I$, constitute a Bayes-Nash cutoff equilibrium if and only if: $\forall i \in I$, if every $j \in I \setminus \{i\}$ participates and bids $\sigma_j(v_j)$ if his value is at least v_j and otherwise does not participate, i maximizes his expected utility by participating and bidding $\sigma_i(v_i)$ if his value is at least v_i and otherwise not participating.*

In the context of the Vickrey auction or any variant that makes agent-independent participation payments (including the RA_ρ auctions), since truthful bidding is a dominant strategy contingent on participation, assuming f has support $[0, 1]$ there is a unique equilibrium of the second-stage game; therefore any

Bayes-Nash cutoff equilibrium can be described simply by the participation cutoff levels. Accordingly, with respect to such auctions we will say things like "value v_0 is a symmetric Bayes-Nash cutoff equilibrium", with truthful bidding in the second stage left implicit. A cutoff equilibrium is *symmetric* if it specifies the same strategy (cutoff level v_0 and bidding strategy) for each agent.

As an example, consider $N = 4$ and values for agents 1, 2, 3, and 4 that are 0.1, 0.6, 0.75, and 0.8, respectively. Imagine, purely hypothetically, that all agents play cutoff level 0.5 and bid truthfully. Then agent 1 does not participate, and agents 2, 3, and 4 place truthful bids and the item is allocated to agent 4. Under the Vickrey auction agent 4 pays 0.75, the second highest bid. Under the redistribution auction RA_1, too, agent 4 makes this payment, but then additional redistribution payments are made. Agent 2 is payed $0.75/3 = 0.25$ and agents 3 and 4 are each payed $0.6/3 = 0.2$. Of the 0.75 "revenue" generated by the Vickrey auction, 0.65 has been redistributed by RA_1. Whether 0.5 or in fact any other cutoff value constitutes an *equilibrium* will depend on the entry costs and the distribution over values, but we will soon see that, given some natural restrictions, a symmetric cutoff equilibrium always exists.

3 Linear cost functions

In this section we provide core extensions of the theory of auctions with entry, going beyond fixed costs and beyond analysis of Vickrey or first-price auctions. We consider the space of cost functions that are linear in agent valuations and have range lying in the open interval bounded by the minimum and maximum possible valuation.

Definition 4 (regular linear cost function)**.** *A cost function $c : [0, 1] \to [0, 1]$ is termed regular linear if and only if $\exists b, k \in [0, 1]$ such that, $\forall v \in [0, 1]$, $c(v) = b + kv \in (0, 1)$.*

The following lemma, which is not specific to *linear* cost functions, will be useful as a first step towards identifying cutoff equilibria.

Lemma 1. *For arbitrary continuous cost function c and values with arbitrary continuous density function with support $[0, 1]$, if cutoff level $v_0 \in (0, 1)$ for each agent is a Bayes-Nash cutoff equilibrium of RA_ρ with reserve price $r \in \Re$, then in that equilibrium any agent with value v_0 has expected utility 0 from participation.*

Proof. Assume the lemma fails to hold for RA_ρ, for arbitrary $\rho \in [0, 1]$ and $r \in \Re$, with cutoff equilibrium v_0. If playing cutoff level v_0 and truthful bidding yields negative expected utility for an agent, clearly setting a higher cutoff (e.g., 1, yielding non-participation and thus utility 0) would be a superior best-response strategy. Alternatively if the strategy yields expected utility $\gamma > 0$, consider the expected utility from participation of an agent with value $v_0 - \epsilon$, for some small $\epsilon > 0$. Noting that the agent will win the auction if and only if no other agent participates, given reserve price r his expected utility equals:

$$R_\rho(v_0, r) + (v_0 - r - \epsilon)G(v_0) - c(v_0 - \epsilon) \tag{5}$$

Regardless of the agent's value, in expectation he will obtain redistribution payment $R_\rho(v_0, r)$ from participation. The difference between the expected utility—under truthful participation—for an agent with value v_0 and for an agent with value $v_0 - \epsilon$ is:

$$[R_\rho(v_0, r) + (v_0 - r)G(v_0) - c(v_0)] - \tag{6}$$
$$[R_\rho(v_0, r) + (v_0 - r - \epsilon)G(v_0) - c(v_0 - \epsilon)] \tag{7}$$
$$= \epsilon G(v_0) - [c(v_0) - c(v_0 - \epsilon)] \tag{8}$$

Because cost function c is continuous, $\epsilon > 0$ can be chosen small enough such that this difference (Eq. (8)) is strictly less than γ. Then the expected utility of participating with value $v_0 - \epsilon$ is greater than 0, contradicting v_0 as a cutoff participation equilibrium since an agent with value $v_0 - \epsilon$ would have positive expected utility from participating. □

A consequence of this lemma is that, for any putative symmetric cutoff equilibrium value v_0 of the RA_ρ auction with reserve price r,

$$R_\rho(v_0, r) + (v_0 - r)G(v_0) = c(v_0) \tag{9}$$

We can now precisely describe conditions for the existence of a symmetric cutoff equilibrium (exceeding the reserve price) for RA_ρ.[8]

Theorem 1. *Consider arbitrary regular linear entry cost function c, values with arbitrary continuous density function with support* $[0, 1]$*, and arbitrary* $r \geq 0$*.* $\forall v_0 \in (r, 1)$*,* v_0 *is a symmetric Bayes-Nash cutoff equilibrium*[9] *of* RA_ρ *with reserve price r if and only if* $R_\rho(v_0, r) + (v_0 - r)G(v_0) = c(v_0)$ *and* $c(0) \geq R_\rho(v_0, r)$*.*

Proof. Consider arbitrary $\rho \in [0, 1]$, $r \geq 0$, regular linear entry cost function $c(v) = b + kv$, $\forall v \in [0, 1]$, and values with arbitrary continuous density function with support $[0, 1]$. For arbitrary $v_0 \in (r, 1)$, note that if $R_\rho(v_0, r) + (v_0 - r)G(v_0) \neq c(v_0)$, then v_0 could not be a cutoff equilibrium, by Lemma 1; and if $c(0) < R_\rho(v_0, r)$, v_0 could not be a cutoff equilibrium because an agent with value 0 would obtain positive utility from participation. Thus, for arbitrary $v_0 \in (r, 1)$, $R_\rho(v_0, r) + (v_0 - r)G(v_0) = c(v_0)$ and $c(0) \geq R_\rho(v_0, r)$ necessarily hold if v_0 is a cutoff equilibrium.

Now assume existence of a $v_0 \in (r, 1)$ satisfying $R_\rho(v_0, r) + (v_0 - r)G(v_0) = c(v_0)$ and $c(0) = b \geq R_\rho(v_0, r)$, and assume all agents other than some $i \in I$ participate and bid truthfully if their value is at least v_0, and otherwise do not participate. If i participates then truthful bidding will be a dominant strategy (by Proposition 1), and so the theorem holds if i obtains non-negative expected net utility from participation if and only if his value is at least v_0. If i's value equals v_0, if he participates his expected utility is:

$$u_i(v_0) = R_\rho(v_0, r) + (v_0 - r)G(v_0) - c(v_0) = 0 \tag{10}$$

[8] In writing the proof for Theorem 1 I was inspired by Matthews [11, Theorem 4.2], who analyzes symmetric equilibria of the Vickrey auction in a constant cost setting.

[9] I.e., the symmetric strategy profile where each agent enters and bids truthfully if his value is at least v_0 and otherwise does not enter is a Bayes-Nash equilibrium.

Bear in mind that $G(v_0) = \frac{c(v_0)-R_\rho(v_0,r)}{v_0-r}$, and that this implies $c(v_0)-R_\rho(v_0,r) > 0$ since $G(v_0) > 0$ and $v_0 - r > 0$. If i's value is $\underline{v} < v_0$ and he participates, his expected utility is:

$$
\begin{aligned}
u_i(\underline{v}) &= R_\rho(v_0,r) + (\underline{v} - r)G(v_0) - c(\underline{v}) && (11)\\
&= R_\rho(v_0,r) + \frac{\underline{v}-r}{v_0-r}(c(v_0) - R_\rho(v_0,r)) - c(\underline{v}) && (12)\\
&\le R_\rho(v_0,r) + \frac{\underline{v}}{v_0}(c(v_0) - R_\rho(v_0,r)) - c(\underline{v}) && (13)\\
&= \frac{\underline{v}}{v_0}(b + kv_0 - R_\rho(v_0,r)) - (b + k\underline{v} - R_\rho(v_0,r)) && (14)\\
&= \frac{\underline{v}}{v_0}(b - R_\rho(v_0,r)) - (b - R_\rho(v_0,r)) \le 0 && (15)
\end{aligned}
$$

We use the fact that $v_0 - r > 0$ and $b = c(0) \ge R_\rho(v_0,r)$. So i cannot gain by participating for any $\underline{v} < v_0$.

Now consider the case where i's value is $\overline{v} > v_0$. Let $p(y)$ equal y if $y \ge v_0$ and r otherwise. If i participates then his expected utility is:

$$
\begin{aligned}
u_i(\overline{v}) &= R_\rho(v_0,r) + \int_0^{\overline{v}} [\overline{v} - p(y)]g(y)\,dy - c(\overline{v}) && (16)\\
&= R_\rho(v_0,r) + \int_0^{v_0} [\overline{v} - p(y)]g(y)\,dy - c(\overline{v}) + \int_{v_0}^{\overline{v}} [\overline{v} - p(y)]g(y)\,dy && (17)\\
&= R_\rho(v_0,r) + (\overline{v} - r)G(v_0) - c(\overline{v}) + \int_{v_0}^{\overline{v}} [\overline{v} - p(y)]g(y)\,dy && (18)\\
&\ge R_\rho(v_0,r) + (\overline{v} - r)G(v_0) - c(\overline{v}) && (19)\\
&= R_\rho(v_0,r) + \frac{\overline{v}-r}{v_0-r}(c(v_0) - R_\rho(v_0,r)) - c(\overline{v}) && (20)\\
&\ge R_\rho(v_0,r) + \frac{\overline{v}}{v_0}(c(v_0) - R_\rho(v_0,r)) - c(\overline{v}) && (21)\\
&= \frac{\overline{v}}{v_0}(b + kv_0 - R_\rho(v_0,r)) - (b + k\overline{v} - R_\rho(v_0,r)) && (22)\\
&= \frac{\overline{v}}{v_0}(b - R_\rho(v_0,r)) - (b - R_\rho(v_0,r)) \ge 0 && (23)
\end{aligned}
$$

Again we use $v_0 - r > 0$ and $b = c(v_0) > R_\rho(v_0,r)$. The move from Eq. (18) to Eq. (19) holds because $p(y) < \overline{v}$ for all $y < \overline{v}$. So i cannot gain from *not* participating when his value is greater than v_0, which completes the proof. □

Since the Vickrey auction is a special case of the RA_ρ auction class with $\rho = 0$, the following is a direct consequence of Theorem 1.

Corollary 1. *Consider arbitrary regular linear entry cost function c, values with arbitrary continuous density function with support $[0,1]$, and arbitrary $r \ge 0$. $\forall v_0 \in (r,1)$, v_0 is a symmetric Bayes-Nash cutoff equilibrium of the Vickrey auction with reserve price r if and only if $(v_0 - r)G(v_0) = c(v_0)$.*

We can now use the combination of Lemma 1 and Theorem 1 to demonstrate that—for reasonable choices of reserve price r—there always exists exactly *one* cutoff equilibrium for the Vickrey auction or a redistribution auction with sufficiently small redistribution parameter. The following lemma will play a significant role, here and in other results to come.

Lemma 2. *Consider arbitrary differentiable function J that is strictly increasing over the interval $[0, 1]$, and arbitrary differentiable function $\mathcal{C}$ that is concave over the interval $[0, 1]$, with $\mathcal{C}(x) \in (0, 1)$, $\forall x \in [0, 1]$. For arbitrary $r \in [0, 1]$, $(v - r)J(v) = \mathcal{C}(v)$ has at most one solution v in the range $(r, 1]$.*

Proof. Consider arbitrary differentiable function J that is strictly increasing over the interval $[0, 1]$, and arbitrary function $\mathcal{C}$ that is concave over the interval $[0, 1]$, with $\mathcal{C}(x) \in (0, 1)$, $\forall x \in [0, 1]$. Consider arbitrary $r \in [0, 1]$. Assume for contradiction that there exist $\underline{v}, \overline{v} \in (r, 1]$ with $\overline{v} > \underline{v}$ such that $(\underline{v} - r)J(\underline{v}) - \mathcal{C}(\underline{v}) = (\overline{v} - r)J(\overline{v}) - \mathcal{C}(\overline{v}) = 0$. This implies:

$$(\overline{v} - r)J(\overline{v}) - (\underline{v} - r)J(\underline{v}) = \mathcal{C}(\overline{v}) - \mathcal{C}(\underline{v}) \tag{24}$$

Noting that $\frac{\partial}{\partial x}[(x - r)J(x)] = J(x) + (x - r)J'(x)$, the left-hand-side can be rewritten $\int_{\underline{v}}^{\overline{v}} [J(x) + (x - r)J'(x)]\, dx$, and the right-hand-side is $\int_{\underline{v}}^{\overline{v}} \mathcal{C}'(x)\, dx$. Thus we have:

$$\int_{\underline{v}}^{\overline{v}} \Big[J(x) + (x - r)J'(x) - \mathcal{C}'(x)\Big]\, dx = 0 \tag{25}$$

Now note that $\mathcal{C}(\underline{v}) - \mathcal{C}(0) = \int_0^{\underline{v}} \mathcal{C}'(x)\, dx$, and thus

$$\begin{aligned} \mathcal{C}(\underline{v}) &= \int_0^{\underline{v}} \mathcal{C}'(x)\, dx + \mathcal{C}(0) && (26)\\ &\geq \int_0^{\underline{v}} \mathcal{C}'(\underline{v})\, dx + \mathcal{C}(0) && (27)\\ &= \underline{v}\mathcal{C}'(\underline{v}) + \mathcal{C}(0) && (28) \end{aligned}$$

The inequality holds because concavity of $\mathcal{C}$ implies $\mathcal{C}'(x) \geq \mathcal{C}'(\underline{v})$, $\forall x \in [0, \underline{v}]$. Then since $\mathcal{C}(\underline{v}) = (\underline{v} - r)J(\underline{v})$ and $\mathcal{C}(0) > 0$, this implies that $J(\underline{v}) > \frac{\underline{v}}{\underline{v} - r}\mathcal{C}'(\underline{v}) \geq \mathcal{C}'(\underline{v})$. By concavity of $\mathcal{C}$, $\mathcal{C}'$ is weakly monotonically decreasing, and J is strictly monotonically increasing by assumption; therefore $\forall x \in [\underline{v}, 1]$, $J(x) - \mathcal{C}'(x) \geq J(\underline{v}) - \mathcal{C}'(\underline{v})$, and thus $J(x) - \mathcal{C}'(x) > 0$. Note also that $(x - r)J'(x) > 0$, $\forall x \in [\underline{v}, 1]$. Therefore we have:

$$\int_{\underline{v}}^{\overline{v}} \Big[J(x) + (x - r)J'(x) - \mathcal{C}'(x)\Big]\, dx > 0 \tag{29}$$

Eqs. (25) and (29) contradict each other, and the lemma follows. □

Theorem 2. *For arbitrary regular linear entry cost function c and values with arbitrary continuous density function with support $[0, 1]$, for arbitrary $r \in [0, 1 - c(1))$, there exists strictly positive ρ^* such that RA_ρ with any $\rho \in [0, \rho^*]$ and reserve price r has a unique symmetric Bayes-Nash cutoff equilibrium.*

Proof. The combination of Lemma 1, Lemma 2, and Theorem 1 brings us most of the way to a proof. We defer the formal proof for now since in Section 4 we prove a stronger result (Theorem 7) that has this theorem as a corollary. □

In practice, the range of ρ values that yield a unique symmetric cutoff equilibrium for RA_ρ will be large, often encompassing the complete $[0, 1]$ interval. The fact that the Vickrey auction has a unique symmetric cutoff equilibrium directly follows from Theorem 2.

Corollary 2. *For arbitrary regular linear entry cost function c and values with arbitrary continuous density function with support $[0, 1]$, the Vickrey auction with reserve price $r \in [0, 1 - c(1))$ has a unique symmetric Bayes-Nash cutoff equilibrium.*

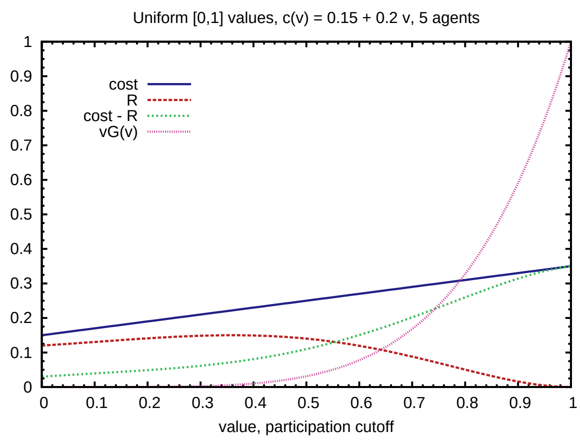

Figure 1: For cost function $c(v) = 0.15 + 0.2v$, with $\rho = 1$ and 5 agents. Note that the cost-minus-redistribution curve intersects the $vG(v)$ curve at exactly one point, which is the cutoff equilibrium value for RA_ρ; this point is lower than the intersection point of $vG(v)$ and the cost curve, which is the cutoff equilibrium value for the Vickrey auction.

Figure 1 illustrates the unique symmetric equilibria of the Vickrey and RA_1 auctions with no reserve price. For the Vickrey auction, it is the point at which the cost line intersects the $vG(v)$ curve; for RA_1 it is somewhat lower, at the point where the cost line minus the redistribution curve intersects the $vG(v)$ curve. Figure 2 illustrates the range of ρ values that maintain a unique symmetric equilibrium, for a setting with 5 agents, uniformly distributed values, and constant costs ($c(v) = x + 0 \cdot v$). The maximum ρ value is plotted as a function of the cost. For most cost levels (all above threshold 0.125 in the example), all $\rho \in [0, 1]$ yield a unique symmetric cutoff equilibrium.

Given that the Vickrey auction and an RA_ρ auction with $\rho > 0$ each have a unique symmetric equilibrium, because RA_ρ yields higher expected utility for every

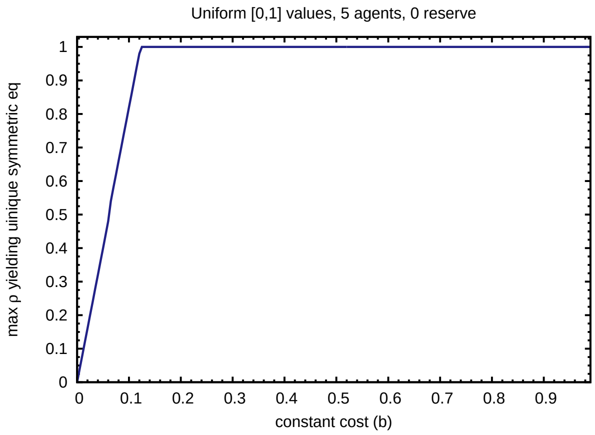

Figure 2: For constant costs, 5 agents, and uniformly distributed values, the maximum ρ yielding a unique symmetric equilibrium is plotted as a function of cost. For costs above 0.125, all ρ values yield a unique symmetric cutoff equilibrium.

agent (regardless of valuation), more agents will participate and a higher expected allocation value will result.

Theorem 3. *For arbitrary regular linear entry cost function and values with arbitrary density function with support* $[0, 1]$*, there exists a* $\rho > 0$ *such that* RA_ρ *with no reserve price yields strictly higher expected allocation value than the Vickrey auction in a unique symmetric Bayes-Nash cutoff equilibrium.*

Proof. For RA_ρ with any positive ρ, if there is a unique symmetric Bayes-Nash cutoff equilibrium, the cutoff value will be lower than that of the Vickrey auction (this can be seen from Eq. (9) combined with the fact that $R_\rho(v_0, 0) > 0, \forall v_0 \in (0, 1), \forall \rho > 0$). By Theorem 2 we know a positive ρ yielding a unique symmetric cutoff equilibrium always exists. Given two mechanisms with unique symmetric cutoff equilibria, any agent that participates in the one with higher cutoff value will participate in the one with lower cutoff, and there exist agents (more precisely, realizations of values) that will participate in the one with lower cutoff but not the one with higher. Moreover, there is positive probability that there will be agents with value above the lower cutoff but none with value above the higher cutoff. Thus, since both the Vickrey auction and the RA_ρ auction choose efficient allocations given the set of participating agents, the expected allocation value of RA_ρ will be strictly higher. □

3.1 Increasing participation through reserve prices

We will now demonstrate a fact that, at first, may seem quite counterintuitive: adding a reserve price to the RA_ρ auction will frequently lead to increased participation. In the Vickrey auction this will never be the case; adding a reserve price there only decreases expected payoff for the agents, since its only effect is to introduce the possibility that agents will pay the entry cost but none will obtain any surplus, or that an agent will win the item but have to pay an increased price. In the case of RA_ρ there is a countervailing factor: a reserve price can provide an agent-independent quasi-guarantee on revenue—if in equilibrium only bidders with value above the reserve price bid, then when there are bidders in equilibrium the revenue will be at least the reserve price. In this way, for the purpose of redistribution, a reserve price plays the role of an "extra bid", so that even in the case where there are just two bidders the reserve provides a basis for redistribution, whereas without it no redistribution could occur without risking a deficit. We now show that imposing a reserve price is very frequently effective at increasing participation, and can always be done in a way that ensures no-deficit in equilibrium.

Theorem 4. *Consider arbitrary regular linear entry cost function c and values with arbitrary continuous density function with support $[0,1]$. Let v_0 be the unique symmetric Bayes-Nash cutoff equilibrium of RA_ρ with no reserve price (where ρ is restricted to choices ensuring that such an equilibrium exists and is unique). There exists an $r > 0$ such that RA_ρ with reserve price r achieves greater expected allocation value in a unique symmetric Bayes-Nash cutoff equilibrium, with no deficit, if $F(v_0) < \frac{N-1}{N-3+\frac{2}{\rho}}$.*

Proof. See Appendix. □

For the standard redistribution auction, RA_1, the condition of Theorem 4 reduces to $F(v_0) < 1$, and so we have the following corollary:

Corollary 3. *For arbitrary regular linear entry cost function c and values with arbitrary continuous density function with support $[0,1]$, if $R_1(v_0, 0) + v_0 G(v_0) = c(v_0)$ has a unique solution $v_0 \in (0,1)$ and $R_1(v_0, 0) \leq c(0)$, there exists an $r > 0$ such that RA_1 with reserve price r yields no deficit and greater expected allocation value than RA_1 with no reserve price in a unique symmetric Bayes-Nash cutoff equilibrium.*

Figure 3 illustrates the effect a reserve price can have on increasing redistribution levels. As the participation cutoff level grows, the amount by which a reserve price increases redistribution also grows. The figure also illustrates how the resulting equilibrium participation level is lowered with a reserve price. Figure 4 illustrates how the allocation value increases (a consequence of lower equilibrium cutoff levels) as a function of reserve price.

3.2 Efficiency of the Vickrey auction

We now shift our focus from allocation value to efficiency, i.e., allocation value minus the aggregate costs of participation. When our goal is maximizing the aggregate social welfare that results from the auction, efficiency is the appropriate metric to consider. Stegeman [17] proved that the Vickrey auction has an efficient equilibrium, which may be asymmetric. Here we will show something different, but highly related: for regular linear entry costs, the unique symmetric equilibrium

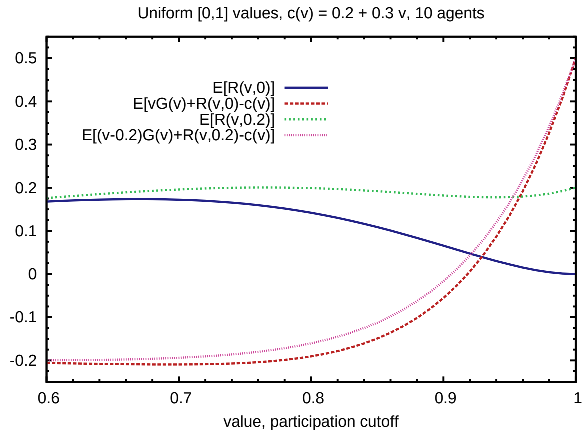

Figure 3: Comparison of redistribution and cutoff equilibrium levels for RA_1 with no reserve price and RA_1 with reserve price of 0.2. The cutoff equilibria can be identified as the points where the bottom two curves cross the 0 line. The cutoff equilibrium for reserve price 0.2 is smaller and thus in equilibrium there will be greater participation and greater expected allocation value.

cutoff strategy of the Vickrey auction is at least as efficient as any other symmetric cutoff strategy.

Theorem 5. *For arbitrary regular linear entry cost function and values with arbitrary density function with support* $[0,1]$*, the cutoff strategy achieved by the Vickrey auction in its unique symmetric Bayes-Nash equilibrium yields strictly greater expected social welfare net of costs than all other symmetric cutoff strategies.*

Proof. Consider arbitrary regular linear cost function c and values with arbitrary density function with support $[0,1]$. Consider the expected social welfare $\mathbb{E}[SW(v)]$ that results as a function of arbitrary symmetric cutoff level v (and truthful bidding) played by all agents. We have:

$$\mathbb{E}[SW(v)] = \int_v^1 NF(x)^{N-1} f(x) x \, dx - N \int_v^1 f(x) c(x) \, dx \tag{30}$$

$$= N \int_v^1 f(x) \left[x F(x)^{N-1} - c(x) \right] dx \tag{31}$$

The first term of Eq. (30) represents the expected allocation value (highest value among participants), and the second term is the aggregate expected entry cost. Examining the partial derivative with respect to the cutoff value, we have:

$$\frac{\partial}{\partial v} \mathbb{E}[SW(v)] = -NF(v)^{N-1} f(v) v + N f(v) c(v) \tag{32}$$

$$= N f(v) \cdot \Big(c(v) - v F(v)^{N-1} \Big) \tag{33}$$

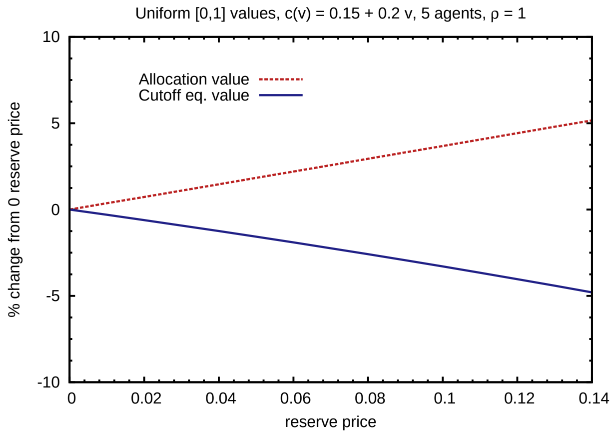

Figure 4: For cost function $c(v) = 0.15 + 0.2v$, with $\rho = 1$ and 5 agents. The cutoff equilibrium value decreases and the allocation value increases linearly as the reserve price increases. At a certain reserve price threshold (0.14, here), a unique equilibrium is no longer sustained without decreasing ρ.

Note that $c(0) > 0$ and $c(1) < 1$ (by regularity of the cost function), $0F(0)^{N-1} = 0$ and $1F(1)^{N-1} = 1$, $vF(v)^{N-1}$ is strictly increasing, and c is concave. Then, by Lemma 2, $c(v)$ and $vF(v)^{N-1}$ intersect at exactly one point v_0 on the interval $[0, 1]$. For all $v \in [0, v_0)$, $c(v) - vF(v)^{N-1} > 0$; and for all $v \in (v_0, 1]$, $c(v) - vF(v)^{N-1} < 0$. Therefore $\mathbb{E}[SW(v)]$ is maximized at v_0, i.e., v_0 is the efficient symmetric cutoff level. By Corollaries 1 and 2, v_0 is precisely the unique symmetric Bayes-Nash cutoff equilibrium of the Vickrey auction with no reserve price. □

In light of this theorem and given our emphasis on analyzing outcomes in unique symmetric equilibria, if we take allocation value net of entry costs as our evaluation metric, then the Vickrey auction is the optimal solution. In the next section we show that this is no longer the case if we consider a richer model of participation costs.

4 Obscurity costs: positive externalities of participation

In the linear cost setting we've been considering, and more generally when an individual's participation costs are independent of the other agents, the Vickrey auction is known to be optimal with respect to maximizing efficiency [16; 17]; it elicits exactly the right amount of participation, so that the marginal expected improvement in allocation value of more participation would not outweigh the added costs. But this model is restrictive, and in particular it does not allow for the possibility that the costs of participating in an auction may depend on the participation strategies of *other* agents. In many cases participation costs will not be purely a function of private type, but rather will incorporate search and logistical costs related, for instance, to finding the auction and going through the process of preparing for participation. It is natural to expect that such costs fall in proportion to the popularity of the auction; in other words, participation may exert positive externalities on other bidders.

Consider this example: a new marketplace is attempting to establish itself as a competitor to the online auction site Ebay. The barriers to participation in the upstart marketplace will be significant as a result of its relative obscurity: becoming aware of the auction, finding out how to sign-up for participation, overcoming psychological concerns about reliability of the new market's privacy and enforcement claims, etc. There is a cost of obscurity. Let oc denote an *obscurity cost* function that maps equilibrium participation rates to (non-negative) costs. If we're concerned with what happens in a unique symmetric cutoff equilibrium, the cutoff value is a completely sufficient stand-in for "equilibrium participation rate"; so $oc(v_0)$ denotes the obscurity cost of an auction to an agent when, amongst other agents, all and only those with value at least v_0 participate. We assume oc is differentiable. Positive participation externalities imply an obscurity cost function $oc(v_0)$ that is increasing in v_0. The obscurity cost borne is independent of the agent's value, but it may come *in addition to* a standard (potentially value-dependent) entry cost function c of the sort we've been considering up until now.

For this enriched model, in Section 4.1 we show that with increasing obscurity costs the Vickrey auction is inefficient and is dominated by an RA_ρ auction. In Section 4.2 we consider the problem of maximally efficient auction design, and in Section 4.3 we outline constraints on achieving optimal efficiency in the context of a mechanism that does not run a deficit.

4.1 Improving efficiency through redistribution

We start with two theorems that directly extend results from Section 3 to this enriched obscurity costs setting. Theorem 6 describes the conditions that hold in a symmetric cutoff equilibrium of RA_ρ, and Theorem 7 demonstrates that, given natural conditions on the cost functions, there is a unique symmetric equilibrium for RA_0 (the Vickrey auction) as well as RA_ρ for a non-empty set of strictly positive values of ρ. Then, in Theorem 8 we demonstrate that a redistribution auction (with $\rho > 0$) is more efficient than the Vickrey auction for a wide space of obscurity cost functions.

Theorem 6. *Consider arbitrary regular linear entry cost function c, arbitrary obscurity cost function oc, values with arbitrary continuous density function with support $[0,1]$, and arbitrary $r \geq 0$. $\forall v_0 \in (r,1)$, v_0 is a symmetric Bayes-Nash cutoff equilibrium of RA_ρ with reserve price r if and only if $R_\rho(v_0,r) + (v_0 - r)G(v_0) = c(v_0) + oc(v_0)$ and $c(0) + oc(v_0) \geq R_\rho(v_0,r)$.*

Proof. See Appendix. □

Theorem 7. *Consider arbitrary regular linear entry cost function c and arbitrary concave obscurity cost function oc such that $c(v)+oc(v') \in (0,1)$, $\forall v, v' \in [0,1]$. For values with arbitrary continuous density function with support $[0,1]$ and arbitrary $r \in [0, 1-c(1)-oc(1))$, there exists strictly positive ρ^* such that RA_ρ with any $\rho \in [0, \rho^*]$ and reserve price r has a unique symmetric Bayes-Nash cutoff equilibrium and in that equilibrium, for arbitrary agent values, no deficit results.*

Proof. Consider arbitrary cost function c and concave obscurity cost function oc such that $c(v) + oc(v') \in (0,1)$, $\forall v, v' \in [0,1]$, values with arbitrary continuous density function with support $[0,1]$, and arbitrary $r \in [0, 1-c(1)-oc(1))$. Choose arbitrary $\rho' \in (0,1]$ such that $R_{\rho'}(v,r) < c(v) + oc(v')$, $\forall v, v' \in [0,1]$. Such a ρ' exists since $c(v) + oc(v') > 0$, $\forall v, v' \in [0,1]$ and $R_\rho(\cdot,\cdot)$ decreases to 0 as ρ approaches 0. Then for arbitrary $\rho \in [0, \rho']$ and $v \in [0,1]$, let $\pi(v) = R_\rho(v,r) + (v-r)G(v) - c(v) - oc(v)$. We have: $\pi(r) = R_\rho(r,r) - c(r) - oc(r) < 0$ and $\pi(1) = R_\rho(1,r) + (1-r)G(1) - c(1) - oc(1) \geq 1 - r - c(1) - oc(1) > 0$; and $\forall v \in [r,1]$, $\pi(v)$ is continuous. Therefore $\pi(v_0) = 0$ at at least one point $v_0 \in (r,1)$, and so a symmetric Bayes-Nash cutoff equilibrium of RA_ρ exists by Theorem 6, and this equilibrium is no-deficit by Proposition 2.

Note that no $v_0 \leq r$ could be a symmetric cutoff equilibrium, because if all agents play cutoff v_0 the expected utility of an agent with value r is $R_{\rho'}(v_0, r) - c(r) - oc(v_0) < 0$, and so an agent with value r would never participate. Note also that cutoff level 1 (full non-participation) is not an equilibrium: letting $x = (1 - c(1) - oc(1)) - r$, observe that $x > 0$. If no other agents participate, a participating agent with value v has expected utility $v - r - c(v) - oc(1) = v - (1 - c(1) - oc(1)) + x - c(v) - oc(1) = x + v - 1 + c(1) - c(v)$. As v approaches 1, $v - 1 + c(1) - c(v)$ approaches 0, and so there exists a v close enough to 1 such that $x + v - 1 + c(1) - c(v) > 0$ and thus participation is a beneficial deviation. Cutoff level 0 (full participation) is also not an equilibrium since $R_\rho(0,r) < c(0) + oc(0)$, so an agent with value 0 obtains negative expected utility from participation.

Now, the argument of Lemma 1 extends naturally to the setting with obscurity costs: any symmetric cutoff equilibrium $v_0 \in (0,1)$ will satisfy $\pi(v_0) = 0$. Thus, given the above considerations, the theorem follows if there is not more than one $v_0 \in (r,1)$ such that $\pi(v_0) = 0$. The proof will follow the path of Lemma 2's proof. For any $v \in [0,1]$, let $\mathcal{C}(v)$ be shorthand for $c(v) + oc(v)$.[10] Assume for contradiction that $\forall \rho^* \in (0, \rho']$, $\exists \rho \in [0, \rho^*]$ such that there exist $\underline{v}, \overline{v} \in (r,1)$ with $\overline{v} > \underline{v}$ such that $\pi(\underline{v}) = \pi(\overline{v}) = 0$, i.e., $R_\rho(\underline{v}, r) + (\underline{v} - r)G(\underline{v}) - \mathcal{C}(\underline{v}) = R_\rho(\overline{v}, r) + (\overline{v} - r)G(\overline{v}) - \mathcal{C}(\overline{v}) = 0$, and thus:

$$(\overline{v} - r)G(\overline{v}) - (\underline{v} - r)G(\underline{v}) = \left[\mathcal{C}(\overline{v}) - R_\rho(\overline{v}, r)\right] - \left[\mathcal{C}(\underline{v}) - R_\rho(\underline{v}, r)\right] \qquad (34)$$

Noting that $\frac{\partial}{\partial x}[(x-r)G(x)] = G(x) + (x-r)g(x)$, the left-hand-side of Eq. (34) can be rewritten as $\int_{\underline{v}}^{\overline{v}} [G(x) + (x-r)g(x)]\, dx$, and the right-hand-side as $\int_{\underline{v}}^{\overline{v}} \mathcal{C}'(x)\, dx +$

[10]Note that $\forall x \in [0,1]$, $\mathcal{C}$ is differentiable at x since c is linear and oc is differentiable (by stipulation in the beginning of Section 4); note also that G is differentiable at x since f is continuous.

$R_\rho(\underline{v}, r) - R_\rho(\overline{v}, r)$. Thus we have:

$$\int_{\underline{v}}^{\overline{v}} \Big[G(x) + (x-r)g(x) - \mathcal{C}'(x)\Big]\, dx + R_\rho(\overline{v}, r) - R_\rho(\underline{v}, r) = 0 \tag{35}$$

Now note that $\mathcal{C}(\underline{v}) - \mathcal{C}(0) = \int_0^{\underline{v}} \mathcal{C}'(x)\, dx$, and thus

$$\mathcal{C}(\underline{v}) = \int_0^{\underline{v}} \mathcal{C}'(x)\, dx + \mathcal{C}(0) \tag{36}$$

$$\geq \int_0^{\underline{v}} \mathcal{C}'(\underline{v})\, dx + \mathcal{C}(0) \tag{37}$$

$$= \underline{v}\mathcal{C}'(\underline{v}) + \mathcal{C}(0) \tag{38}$$

The inequality holds by concavity of $\mathcal{C}$. Then since $\mathcal{C}(\underline{v}) = (\underline{v} - r)G(\underline{v}) + R_\rho(\underline{v}, r)$ and $\mathcal{C}(0) > 0$, this implies that $G(\underline{v}) > \frac{\underline{v}}{\underline{v}-r}\mathcal{C}'(\underline{v}) - \frac{1}{\underline{v}-r}R_\rho(\underline{v}, r) \geq \mathcal{C}'(\underline{v}) - \frac{1}{\underline{v}-r}R_\rho(\underline{v}, r)$. G is monotonically strictly increasing, and by concavity of $\mathcal{C}$, $\mathcal{C}'$ is monotonically weakly decreasing; therefore $\forall x \in [\underline{v}, 1]$, $G(x) - \mathcal{C}'(x) \geq G(\underline{v}) - \mathcal{C}'(\underline{v})$, and thus:

$$G(x) - \mathcal{C}'(x) > -\frac{1}{\underline{v} - r} R_\rho(\underline{v}, r) \tag{39}$$

Let $\gamma = \int_{\underline{v}}^{\overline{v}} (x - r)g(x)\, dx$, and note that $\gamma > 0$ since $g(x) > 0$, $\forall x \in [0, 1]$. Returning to Eq. (35) and substituting, we have:

$$0 = \int_{\underline{v}}^{\overline{v}} \Big[G(x) + (x-r)g(x) - \mathcal{C}'(x)\Big]\, dx + R_\rho(\overline{v}, r) - R_\rho(\underline{v}, r) \tag{40}$$

$$> \gamma - \int_{\underline{v}}^{\overline{v}} \frac{1}{\underline{v} - r} R_\rho(\underline{v}, r)\, dx + R_\rho(\overline{v}, r) - R_\rho(\underline{v}, r) \tag{41}$$

$$= \gamma - \frac{\overline{v} - \underline{v}}{\underline{v} - r} R_\rho(\underline{v}, r) + R_\rho(\overline{v}, r) - R_\rho(\underline{v}, r) \tag{42}$$

$$= \gamma - \Big(\frac{\overline{v} - r}{\underline{v} - r} R_\rho(\underline{v}, r) - R_\rho(\overline{v}, r)\Big) \tag{43}$$

But for small enough $\rho^* > 0$, $\forall \rho \in [0, \rho^*]$, $\frac{\overline{v}-r}{\underline{v}-r} R_\rho(\underline{v}, r) - R_\rho(\overline{v}, r) < \gamma$ (as ρ approaches 0, $\forall x \in [0, 1]$, $R_\rho(x, r)$ approaches 0). We have a contradiction, and so $\exists \rho^* \in (0, \rho']$ such that $\forall \rho \in [0, \rho^*]$, there is a unique $v_0 \in [0, 1]$ that is a Bayes-Nash cutoff equilibrium of RA_ρ with reserve price r. □

Since the Vickrey auction is a special case of the RA_ρ auction class, for concave obscurity costs[11] Theorem 7 tells us that the Vickrey auction as well as the redistribution auction, for some space of redistribution parameters ρ, each have unique symmetric cutoff equilibria. This sets up a natural comparison. We can now address the question of whether redistribution is helpful to efficiency in an obscurity costs setting; Theorem 8 demonstrates that it is.

[11]The necessary restriction to *concave* obscurity costs for uniqueness of the equilibrium is unfortunate; in cases where obscurity costs are non-linear, it seems more natural to expect them to be convex, corresponding to diminishing marginal positive participation externalities as the participation rate grows.

Theorem 8. *Consider arbitrary regular linear entry cost function c and arbitrary concave strictly increasing obscurity cost function oc such that $c(v) + oc(v') \in [0,1]$, $\forall v, v' \in [0,1]$. For values with arbitrary density function with support $[0,1]$, there exists a $\rho > 0$ such that RA_ρ has a unique symmetric Bayes-Nash cutoff equilibrium and it yields greater expected social welfare than the the unique symmetric Bayes-Nash cutoff equilibrium of the Vickrey auction.*

Proof. Consider arbitrary regular linear entry cost function c and arbitrary concave strictly increasing obscurity cost function oc such that $c(v) + oc(v') \in [0,1]$, $\forall v, v' \in [0,1]$, and values with arbitrary density function with support $[0,1]$. We will show that there exists a $\rho > 0$ such that RA_ρ has a unique symmetric cutoff equilibrium $\underline{v}$, that it is lower than the unique Vickrey auction cutoff equilibrium (call it $\overline{v}$), and that expected allocation value minus costs (efficiency) is greater with cutoff $\underline{v}$ than with $\overline{v}$.

The expected efficiency under arbitrary symmetric cutoff strategy $v_0 \in [0,1]$ equals the efficiency that would result if there were no obscurity costs minus: $N \cdot oc(v_0) \cdot \int_{v_0}^1 f(x)\,dx$. So expected efficiency in equilibrium v_0, given N agents and cost functions c and oc, equals:

$$\int_{v_0}^1 NF(x)^{N-1} f(x) x\,dx - N\int_{v_0}^1 f(x)c(x)\,dx - Noc(v_0)\int_{v_0}^1 f(x)\,dx \tag{44}$$

$$= N\int_{v_0}^1 f(x)\big[xF(x)^{N-1} - c(x) - oc(v_0)\big]\,dx \tag{45}$$

The first term of Eq. (44) represents the expected allocation value (highest value among participants), and the second and third combined are the aggregate expected entry costs. Examining the partial derivative with respect to the cutoff value, we get:

$$\frac{\partial}{\partial v_0}\left(\int_{v_0}^1 NF(x)^{N-1} f(x) x\,dx - N\int_{v_0}^1 f(x)c(x)\,dx - Noc(v_0)\int_{v_0}^1 f(x)\,dx\right) \tag{46}$$

$$= -NF(v_0)^{N-1} f(v_0) v_0 + Nf(v_0)c(v_0) - Noc'(v_0)\int_{v_0}^1 f(x)\,dx + Nf(v_0)oc(v_0) \tag{47}$$

$$= Nf(v_0)\cdot\left(c(v_0) + oc(v_0) - v_0F(v_0)^{N-1} - oc'(v_0)\frac{1-F(v_0)}{f(v_0)}\right) \tag{48}$$

For the cutoff equilibrium $\overline{v}$ of the Vickrey auction (which is unique, by Theorem 7), we know from an argument completely analogous to Lemma 1 that $\overline{v}F(\overline{v})^{N-1} = c(\overline{v}) + oc(\overline{v})$. So, evaluating the partial derivative at $v_0 = \overline{v}$, we have:

$$Nf(\overline{v})\cdot\left(c(\overline{v}) + oc(\overline{v}) - \overline{v}F(\overline{v})^{N-1} - oc'(\overline{v})\frac{1-F(\overline{v})}{f(\overline{v})}\right) \tag{49}$$

$$= -Nf(\overline{v})\cdot oc'(\overline{v})\frac{1-F(\overline{v})}{f(\overline{v})} \tag{50}$$

$$= -N\cdot oc'(\overline{v})\cdot(1-F(\overline{v})) \tag{51}$$

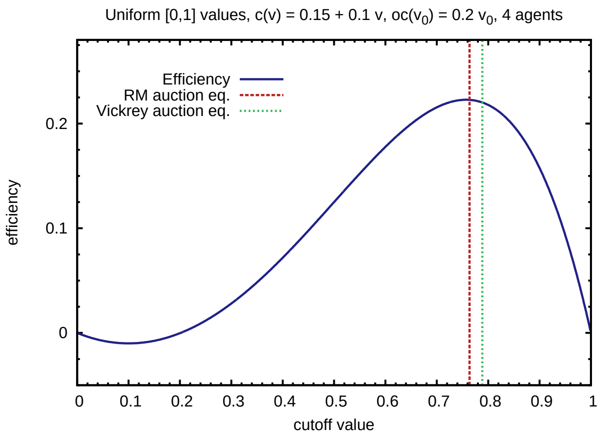

Figure 5: Plot of the expected efficiency (allocation value minus cost) that will result, as a function of symmetric cutoff value played by all agents. For cost functions $c(v) = 0.15 + 0.1v$ and $oc(v_0) = 0.2v_0$, with 4 agents. The RA_1 auction equilibrium cutoff level is significantly lower than that of the Vickrey auction, which, in this obscurity costs setting, makes it more efficient.

N is positive, $1-F(\overline{v})$ is positive, and, given the assumption that obscurity costs are strictly increasing in cutoff value, $oc'(\overline{v})$ is positive; thus the social welfare derivative at $\overline{v}$ is negative, indicating that there is an $\epsilon > 0$ such that any cutoff in the interval $[\overline{v} - \epsilon, \overline{v})$ yields greater efficiency than $\overline{v}$.

It only remains to show that RA_ρ for some $\rho \in (0, 1]$ achieves unique symmetric cutoff equilibrium $v_0 \in [\overline{v} - \epsilon, \overline{v})$. Note that for any $\delta^* > 0$, we can select $\rho \in (0, 1]$ such that $R_\rho(v, 0) \in (0, \delta^*)$, $\forall v \in [0, 1)$. Since values have density function with support $[0, 1]$ and $c(v) + oc(v)$ is concave, continuous, and $\in (0, 1)$, $\forall v \in [0, 1]$, the value v'_0 that solves $v'_0 G(v'_0) = c(v'_0) + oc(v'_0) - \delta$ is strictly less than $\overline{v}$ for any $\delta \in (0, c(0) + oc(0))$ and converges to $\overline{v}$ in the limit as δ goes to 0. Choose $\delta^* \in (0, c(0)+oc(0))$ such that the solution to $v'_0 G(v'_0) = c(v'_0)+oc(v'_0)-\delta^*$ is in the interval $[\overline{v} - \epsilon, \overline{v})$ and choose ρ such that RA_ρ has a unique symmetric Bayes-Nash cutoff equilibrium and $R_\rho(v, 0) \leq \delta^*$, $\forall v \in [0, 1)$ (such a ρ exists by Theorem 7). Then the cutoff equilibrium of RA_ρ is $\underline{v}$, the solution to $\underline{v}G(\underline{v}) = c(\underline{v}) + oc(\underline{v}) - R_\rho(\underline{v}, 0)$, which is in the interval $[\overline{v} - \epsilon, \overline{v})$. This completes the proof. □

Figure 5 illustrates the efficiency gain—in the unique symmetric Bayes-Nash cutoff equilibria—of RA_1 over the Vickrey auction for one particular linear cost function and linear obscurity cost function. The Vickrey auction equilibrium is to the right of the peak of the efficiency curve (under-participation), and the RA_1 equilibrium moves closer to the peak, increasing efficiency through increased participation.

4.2 Welfare-optimal auction design with obscurity costs

The preceding analysis demonstrates that we can always improve the Vickrey auction outcome by moving to a redistribution auction when there are increasing (concave) obscurity costs. In this subsection we go further and consider the following more general mechanism design question: in a setting with linear entry costs and obscurity costs, given a restriction to auctions that allocate the good to the agent with highest value among the participants and are dominant strategy incentive compatible, which auction maximizes efficiency? We will continue to focus on *symmetric* cutoff equilibria, which motivates the following optimization criterion:

Definition 5 (welfare-optimal auction in auction space S)**.** *A welfare-optimal auction in auction space S is a member of S with a symmetric Bayes-Nash cutoff equilibrium that yields at least as much expected efficiency as any symmetric Bayes-Nash cutoff equilibrium of any auction in S.*

This definition gives us the flexibility to consider, e.g., a welfare-optimal auction in the space of *all* allocatively-efficient auctions, or a welfare-optimal auction among allocatively-efficient auctions that have a unique symmetric Bayes-Nash cutoff equilibrium, etc. When we take the space of auctions with a *unique* symmetric equilibrium as our optimization space, welfare-optimality is particularly meaningful, since the comparison between auctions becomes very natural; when the equilibrium is not unique there is an equilibrium selection problem and a welfare-optimal auction may have one symmetric equilibrium that is less efficient than an equilibrium of some other auction in the space. Note that for arbitrary regular linear entry costs and concave obscurity costs, there always exists a welfare-optimal allocatively-efficient auction—even with the added restriction to the space of auctions with a *unique* symmetric cutoff equilibrium—since Theorem 7 demonstrates that the Vickrey auction and also a space of RA_ρ auctions have a unique symmetric cutoff equilibrium (so set S is non-empty).

Restricting ourselves to allocatively-efficient auctions, the search for a welfare-optimal auction (with or without further restrictions) is narrowed significantly by the following result which follows from Holmstrom [8], stating that any allocatively-efficient auction is a Vickrey auction where each agent's payment is modified by an amount independent of his bid.[12] We will call this class of auctions the *Groves auctions* (see [7]), and call a Groves auction's Vickrey-modifying payments the *participation payments.*

Proposition 3 ([8])**.** *An auction is allocatively-efficient (dominant strategy incentive compatible, allocating the good to the highest bidder) if and only if it is a Groves auction—i.e., it charges the highest bidder the second highest bid, and additionally pays each agent a quantity that is independent of his own bid.*

[12] The characterization of efficient and strategyproof mechanisms as the Groves class of mechanisms was first proved by Green and Laffont [5] for the *unrestricted* typespace; Holmstrom took the result significantly further by showing it to hold even in highly structured domains such as single-item allocation.

Lemma 3. *For arbitrary continuous cost function c and obscurity cost function oc, and values with arbitrary continuous density function with support* $[0, 1]$*, if* $v_0 \in (0, 1)$ *is a symmetric Bayes-Nash cutoff equilibrium of a Groves auction, in that equilibrium the expected participation payment received by each bidder equals* $c(v_0) + oc(v_0) - v_0 G(v_0)$*.*

Proof. See Appendix. □

One implication of this lemma (combined with Proposition 3) is that if an allocatively-efficient auction has a symmetric Bayes-Nash cutoff equilibrium, then it is a Groves auction in which—in that equilibrium—each participating agent has the same expected participation payment, given the distribution over others' values. We say *expected* participation payment because certain Groves auctions, including the redistribution auctions RA_ρ, define Vickrey-auction-modifying payments for each agent that are not *fixed*, but rather depend on the bids of other agents and thus may vary across the agents. However, if a Groves auction has symmetric Bayes-Nash cutoff equilibrium v_0 and in that equilibrium each bidder's expected participation payment is X, then the Groves auction that pays each participant a *fixed* payment X regardless of others' bids will also have v_0 as a symmetric cutoff equilibrium. The following lemma follows virtually immediately from the fact that each agent participates if and only if his *expected* utility from doing so is at least 0.

Lemma 4. *For arbitrary continuous cost function c, obscurity cost function oc, and values with arbitrary continuous density function with support* $[0, 1]$*, if cutoff* $v_0 \in (0, 1)$ *is a symmetric Bayes-Nash cutoff equilibrium of a Groves auction that makes bid-dependent participation payments, and in that equilibrium the expected participation payment received by each bidder is* X*, then the Groves auction that pays each agent a fixed participation payment* X *also has* v_0 *as a symmetric Bayes-Nash cutoff equilibrium. In symmetric cutoff equilibrium* v_0 *both auctions yield the same expected utility to each agent and the same expected revenue.*

Therefore, for the purposes of this section in which we seek a welfare-optimal allocatively-efficient auction, it is sufficient to restrict our attention to Groves auctions with fixed participation payments; a welfare-optimal allocatively-efficient auction can always be found within this class. Going forward, we will use notation Groves(X) to denote a Groves auction that makes fixed participation payment X to each bidder.

Before mapping out a method for identifying a welfare-optimal auction, with the following result we completely characterize the symmetric cutoff equilibria of fixed participation payment Groves auctions.

Proposition 4. *Consider arbitrary regular linear entry cost function c and obscurity cost function oc such that* $c(v) + oc(v) \in (0, 1)$*,* $\forall v \in [0, 1]$*, values with arbitrary continuous density function with support* $[0, 1]$*, and arbitrary* $X \in \Re$*. The following is an exhaustive characterization of the symmetric Bayes-Nash cutoff equilibria of the Groves(*X*) auction:*

- $\forall v_0 \in (0,1)$, *if* $X = c(v_0) + oc(v_0) - v_0 G(v_0)$ *and* $X \leq c(0) + oc(v_0)$, *then* v_0 *is a symmetric Bayes-Nash cutoff equilibrium.*

- *If* $X \geq c(v) + oc(0) - (N-1)\int_0^v F(x)^{N-2} f(x)(v-x)\,dx$, $\forall v \in [0,1]$, *full-participation is a symmetric Bayes-Nash cutoff equilibrium.*

If oc is concave and non-decreasing and $vG(v)$ *is convex over* $[0,1]$*: if* $0 \leq X < c(0) + oc(0)$*, there is a unique symmetric cutoff equilibrium and it is interior (i.e.,* $\in (0,1)$*); if* $X \geq c(0) + oc(0)$*, there may be between 0 and 3 symmetric cutoff equilibria, with a maximum of two that are interior.*

Proof. See Appendix. □

For concave obscurity costs and convex $vG(v)$, considering the range of possible values for participation payment X, at $X = 0$ (the Vickrey auction) there is a unique symmetric cutoff equilibrium; as we increase X a second interior symmetric cutoff equilibrium may emerge, and full-participation may also emerge as an equilibrium; when X is large enough the interior equilibria vanish and there will either be no symmetric cutoff equilibrium or, if X is sufficiently large, full-participation.

We now demonstrate characteristics of a welfare-optimal auction which directly imply a method for identifying such. Recall that our restriction to auctions that make fixed participation payments is without loss, with respect to the goal of finding a welfare-optimal auction: it is not possible for an auction that does not make fixed participation payments to yield greater expected efficiency than all that do.

Theorem 9. *For arbitrary regular linear entry cost function c and obscurity cost function oc such that* $c(v) + oc(v') \in (0,1)$, $\forall v, v' \in [0,1]$*, for values with arbitrary continuous density function with support* $[0,1]$*, among all allocatively-efficient auctions that make fixed participation payments, a welfare-optimal auction is a Groves*(X) *auction where* X *satisfies one of the following:*

- $X = c(v_0) + oc(v_0) - v_0 F(v_0)^{N-1}$*, where* v_0 *solves*

$$c(v_0) + oc(v_0) - v_0 F(v_0)^{N-1} = oc'(v_0)\frac{1 - F(v_0)}{f(v_0)} \tag{52}$$

- $X = c(0) + oc(F^{-1}(k^{\frac{1}{N-1}}))$*, where k is the slope of the linear cost function*

- $X \geq \max_{v \in [0,1]}[c(v) + oc(0)]$

Proof. Consider arbitrary regular linear entry cost function c and obscurity cost function oc such that $c(v) + oc(v') \in (0,1)$, $\forall v, v' \in [0,1]$, and values with arbitrary continuous density function with support $[0,1]$. Consider any allocatively-efficient auction that makes fixed participation payments. From Proposition 3 we know the auction is a Groves(X) auction, for some $X \in \Re$. By Lemma 3, for every $v_0 \in (0,1)$, there is at most one $X \in \Re$ such that Groves(X) has v_0 as a symmetric Bayes-Nash cutoff equilibrium; if Groves(X) has $v_0 \in (0,1)$ as a symmetric Bayes-Nash cutoff equilibrium, then $X = c(v_0) + oc(v_0) - v_0 F(v_0)^{N-1}$. So, for arbitrary $v_0 \in (0,1)$, let:

$$X_{v_0} = c(v_0) + oc(v_0) - v_0 F(v_0)^{N-1} \tag{53}$$

We can cast the identification of a welfare-optimal allocatively-efficient auction as an optimization over the set of values $V \subseteq (0,1)$ that are symmetric Bayes-Nash cutoff equilibria of some allocatively-efficient auction, which by Proposition 4 are the $v_0 \in (0,1)$ for which $X_{v_0} \leq c(0) + oc(v_0)$, i.e., for which:

$$c(v_0) + oc(v_0) - v_0 F(v_0)^{N-1} \leq c(0) + oc(v_0) \tag{54}$$

A welfare-optimal allocatively-efficient auction will then either be Groves(X_{v_0}) for the social-welfare maximizing choice $v_0 \in V$, or else a Groves auction that elicits full-participation.

Following the analysis in the proof of Theorem 8, if symmetric cutoff strategy v_0 is played then the expected social welfare will be:

$$\int_{v_0}^{1} NF(x)^{N-1} f(x) x \, dx - N \int_{v_0}^{1} f(x) c(x) \, dx - Noc(v_0) \int_{v_0}^{1} f(x) \, dx \tag{55}$$

Examining the partial derivative with respect to the cutoff value, we get:

$$\frac{\partial}{\partial v_0} \left(\int_{v_0}^{1} NF(x)^{N-1} f(x) x \, dx - N \int_{v_0}^{1} f(x) c(x) \, dx - Noc(v_0) \int_{v_0}^{1} f(x) \, dx \right) \tag{56}$$

$$= Nf(v_0) \cdot \left(c(v_0) + oc(v_0) - v_0 F(v_0)^{N-1} - oc'(v_0) \frac{1 - F(v_0)}{f(v_0)} \right) \tag{57}$$

$$= Nf(v_0) \cdot \left(X_{v_0} - oc'(v_0) \frac{1 - F(v_0)}{f(v_0)} \right) \tag{58}$$

This quantity is negative whenever $X_{v_0} < oc'(v_0) \frac{1-F(v_0)}{f(v_0)}$, implying that there is a social-welfare-improving *lower* cutoff level (there is under-participation at v_0); it is positive whenever $X_{v_0} > oc'(v_0) \frac{1-F(v_0)}{f(v_0)}$, implying that there is a social-welfare-improving *higher* cutoff level (there is over-participation at v_0).

Since $c(v) - c(0)$ is concave in v, by Lemma 2 $c(0) + oc(v)$ and X_v intersect at no more than one point on the interval $(0,1)$,[13] and $X_1 < c(0) + oc(1)$. Therefore if some v is in V, then all $v' \in (v,1)$ are also in V; likewise if some v is not in V, then all $v' \in (0,v)$ are also not in V. Therefore if for some $v_0 \in V$, $X_{v_0} > oc'(v_0) \frac{1-F(v_0)}{f(v_0)}$, then the auction can be improved by moving to a Groves($X_{v'}$) auction for some $v' > v_0$ and so Groves(X_{v_0}) could not be optimal. If $X_{v_0} < oc'(v_0) \frac{1-F(v_0)}{f(v_0)}$, Groves($X_{v_0}$) may be optimal only in the case that for all $v \in (0, v_0)$, $v \notin V$, i.e., only if $X_{v_0} = c(0) + oc(v_0)$. $X_{v_0} = c(0) + oc(v_0)$ combined with Eq. (53) entails that $F(v_0)^{N-1} = k$

[13] At an intersection point we have $X_v - c(0) - oc(v) = 0$, i.e., $c(v) + oc(v) - vG(v) - c(0) - oc(v) = 0$, i.e., $c(v) - c(0) = vG(v)$. Lemma 2 then applies directly to show a maximum of one intersection.

for k that is the slope of the linear cost function ($c(v) = b + kv$, $\forall v \in [0, 1]$), and thus $X_{v_0} = c(0) + oc(F^{-1}(k^{\frac{1}{N-1}}))$.

Finally, if full-participation yields higher social welfare than any interior symmetric cutoff equilibrium level, then achieving this with arbitrary participation payment $X \geq \max_{v\in[0,1]}[c(v) + oc(0)]$ will be optimal. □

This theorem immediately suggests the following algorithm for identifying a welfare-optimal allocatively-efficient auction:

i) determine the set V of all $v \in (0, 1)$ that satisfy:

$$c(v) + oc(v) - vF(v)^{N-1} = oc'(v)\frac{1 - F(v)}{f(v)} \leq c(0) + oc(v) \tag{59}$$

ii) add value $c(0) + oc(F^{-1}(k^{\frac{1}{N-1}}))$ to set V (where $c(v) = b + kv$, $\forall v \in [0, 1]$)

iii) evaluate each element of V according to expected social welfare, determining:

$$v^* = \underset{v_0\in V\cup\{0\}}{\arg\max} \int_{v_0}^{1} f(x)\left[xF(x)^{N-1} - c(x) - oc(v_0)\right] dx \tag{60}$$

A welfare-optimal allocatively-efficient auction is the Vickrey auction modified to pay each participating agent a bonus of $c(v^*) + oc(v^*) - v^*F(v^*)^{N-1}$ if $v^* \in V$ and $\max_{v\in[0,1]}[c(v) + oc(0)]$ (i.e., $oc(0) + \max\{c(0), c(1)\}$) if $v^* = 0$.

We can naturally say more for specific value distributions. When values are uniformly distributed and obscurity costs are linear, for instance, the welfare-optimal allocatively-efficient auction can be precisely pinned down as follows.

Theorem 10. *Consider values uniformly distributed over $[0, 1]$, and arbitrary regular linear entry cost function c and linear obscurity cost function oc such that $c(v) + oc(v') \in (0, 1)$, $\forall v, v' \in [0, 1]$. If $\exists v \in (0, 1)$ such that $c(v) + oc(v) - v^N = oc'(v)(1 - v)$, let $v_0 = \max\{v \in (0, 1) \,|\, c(v) + oc(v) - v^N = oc'(v)(1 - v)\}$ and $v^* = \min\{v \in [v_0, 1) \,|\, c(v) - c(0) \leq v^N\}$; among all allocatively-efficient auctions, letting $X = c(v^*) + oc(v^*) - v^{*N}$, the Groves$(X)$ auction is welfare-optimal if:*

$$oc(0) - (1 - v^*)oc(v^*) > v^*c(v^*/2) + \frac{v^{*N+1}}{N + 1} \tag{61}$$

Otherwise, or if $\nexists v \in (0, 1)$ such that $c(v)+oc(v)-v^N = oc'(v)(1-v)$, the Groves$(Y)$ auction is welfare-optimal where $Y = \max_{v\in[0,1]}[c(v) + oc(0)]$.

Proof. See Appendix. □

4.3 Full efficiency without a deficit

Theorem 8 demonstrates that when there are obscurity costs we can improve the Vickrey auction outcome by moving to a redistribution auction, which never yields a budget deficit. Theorem 9 describes how to achieve optimal efficiency, in the absence of any revenue or deficit constraints. Now we ask whether optimal efficiency can always be achieved with an allocatively-efficient and ex post no-deficit auction, i.e., one that never defines aggregate payments that exceed 0. Theorem 11, below, answers this question negatively. Note that while the class of auctions that make *fixed* participation payments sufficed for the analysis of the last subsection, to satisfy ex post no-deficit we must move beyond: any allocatively-efficient auction that pays an agent i some positive value unconditionally will run a deficit, for instance, if i is the only bidder.

When the Vickrey auction is inefficient due to underparticipation, to achieve efficiency we seek to increase agents' incentives to participate. If, in every such scenario, there is a redistribution auction that returns enough Vickrey auction revenue to the agents such that the efficient cutoff level is achieved in equilibrium, then a no-deficit and efficient auction always exists. However, the converse does not follow. While RA_1 comes close to redistributing all revenue, and thus comes commensurately close to maximizing participation without running a deficit, in some cases another Groves auction outside of this class will return even more revenue without running a deficit.[14] Which ex post no-deficit Groves auction elicits the greatest participation level will heavily depend on the specific distribution of agent values. However, [4] identifies a useful distribution-independent upper-bound on feasible redistribution: in any dominant strategy efficient and ex post no-deficit auction, no agent's redistribution payment exceeds the second highest bid amongst the *other* agents. We use this fact to demonstrate that a no-deficit and efficient auction does not always exist when there are obscurity costs.

Theorem 11. *Consider arbitrary regular linear entry cost function c and obscurity cost function oc such that $c(v)+oc(v) \in (0,1)$, $\forall v \in [0,1]$, and values with arbitrary continuous density function with support $[0,1]$ such that the efficient symmetric cut-off level $v^* \in (0,1)$ (i.e., is interior). There does not exist an ex post no-deficit and allocatively-efficient auction in which v^* is a symmetric Bayes-Nash cutoff equilibrium if the derivative of the obscurity cost function at v^* exceeds the hazard rate at v^* times the expected second high bid amongst $N-1$ agents; i.e., if:*[15]

$$oc'(v^*) > \frac{f(v^*)}{1-F(v^*)} \int_{v^*}^{1} h^2_{N-1}(x)x\,dx \tag{62}$$

Proof. Consider arbitrary regular linear entry cost function c and obscurity cost function oc such that $c(v) + oc(v) \in (0,1)$, $\forall v \in [0,1]$, and values with arbitrary

[14] It can be show that no dominant strategy efficient and ex post no-deficit mechanism is *dominant* with respect to revenue-minimization, yielding no more revenue than any other such mechanism on every profile of bids.

[15] Recall our notational shorthand of h^k_m to denote the probability density function for the k^{th} highest among m values drawn identically and independently from f. Thus, $h^2_{N-1}(x) = (N-1)(N-2)F(x)^{N-3}(1-F(x))f(x)$.

continuous density function with support $[0, 1]$ such that the efficient symmetric cutoff level v^* is $\in (0, 1)$. Consider an allocatively-efficient auction (i.e., any Groves auction) that is ex post no-deficit and has v^* as a symmetric Bayes-Nash cutoff equilibrium; by Lemma 3 we know that in cutoff equilibrium v^* each participant's expected participation payment X equals $c(v^*) + oc(v^*) - vF(v^*)^{N-1}$. Moreover, the proof of Theorem 9 demonstrates[16] that:

$$X = c(v^*) + oc(v^*) - vF(v^*)^{N-1} = oc'(v^*)\frac{1 - F(v^*)}{f(v^*)} \quad (63)$$

From [4][17] we know that the participation payment for any agent in a no-deficit and allocatively-efficient auction is no greater than the second highest bid amongst the other agents, and thus also the *expected* participation payment is no greater than the expected second highest bid amongst the others. We have:

$$X \leq \int_{v^*}^{1} h_{N-1}^2(x)x\,dx \quad (64)$$

Putting Eqs. (63) and (64) together, we have:

$$oc'(v^*) \leq \frac{f(v^*)}{1 - F(v^*)} \int_{v^*}^{1} h_{N-1}^2(x)x\,dx \quad (65)$$

Thus if $oc'(v^*) > \frac{f(v^*)}{1-F(v^*)} \int_v^1 h_{N-1}^2(x)x\,dx$, the efficient cutoff equilibrium v^* is not achievable by an ex post no-deficit and allocatively-efficient auction. □

As a demonstration that the conditions of the theorem are met by natural value distributions and cost functions of the sort we are concerned with, consider:

> **Example 1:** There are 3 agents and values are uniformly distributed over $[0, 1]$. There is a fixed entry cost $c(v) = 0.25$ for all values $v \in [0, 1]$, and there are linear obscurity costs $oc(v_0) = 0.6v_0$, $\forall v_0 \in [0, 1]$.

In this example, if symmetric cutoff level v_0 is played, every agent's entry cost will be the same: $0.25 + 0.6v_0$. One can verify that the optimal symmetric cutoff level v^* is approximately 0.9. The hazard rate at v^* is $\frac{f(v^*)}{1-F(v^*)} = 10$, and the expected second-highest bid amongst 2 agents, each of whom only bids if his value exceeds 0.9, equals 0.0092. The theorem here states that v^* is not achievable by a no-deficit and allocatively-efficient auction, since $oc'(v^*) = 0.6 > 10 \cdot 0.0092$.

If we weaken the no-deficit constraint from *ex post* to *ex ante*, so that we are willing to tolerate negative revenue as long as revenue is non-negative in expectation,

[16] Eq. (63) is always satisfied at an efficient interior symmetric cutoff *level*; that level may or may not be achievable in equilibrium, and Theorem 9 goes further to analyze what auction would be welfare-optimal in the case that it is not.

[17] Specifically, see [4, Lemma 3.1], the proof of which can be summarized as follows: i) an agent's participation payment must be independent of his bid; ii) the greatest lower-bound on revenue that can be established independent of the agent's bid is the second-highest bid amongst the other agents; and iii) if we pay the agent more than the revenue that can be guaranteed independent of his bid, a deficit may result.

achieving the efficient equilibrium becomes possible in many more cases, including in the example described above. The following theorem precisely characterizes the conditions under which an ex ante no-deficit and allocatively-efficient auction exists that achieves the efficient symmetric cutoff level in equilibrium.

Theorem 12. *Consider arbitrary regular linear entry cost function c and obscurity cost function oc such that $c(v)+oc(v) \in (0,1)$, $\forall v \in [0,1]$, and values with arbitrary continuous density function with support $[0,1]$. If the efficient symmetric cutoff level $v^* \in (0,1)$ (i.e., is interior), there exists an ex ante no-deficit and allocatively-efficient auction in which v^* is a symmetric Bayes-Nash cutoff equilibrium if and only if $c(v^*) - c(0) \leq v^* F(v^*)^{N-1}$ and:*

$$oc'(v^*) \leq \frac{f(v^*)}{N(1-F(v^*))^2} \int_{v^*}^{1} h_N^2(x) x \, dx \tag{66}$$

If the efficient symmetric cutoff level is $v^* = 0$ (i.e., full-participation), there exists an ex ante no-deficit and allocatively-efficient auction achieving it if and only if:

$$N\big(oc(0) + \max\{c(0), c(1)\}\big) \leq \int_{0}^{1} h_N^2(x) x \, dx \tag{67}$$

Proof. Consider arbitrary regular linear entry cost function c and obscurity cost function oc such that $c(v) + oc(v) \in (0,1)$, $\forall v \in [0,1]$, and values with arbitrary continuous density function with support $[0,1]$. First consider the case where the efficient symmetric cutoff level v^* is $\in (0,1)$. If an allocatively-efficient auction (i.e., any Groves auction) has v^* as a symmetric Bayes-Nash cutoff equilibrium, then by Lemma 3 we know that in cutoff equilibrium v^* each participant's expected participation payment X equals $c(v^*) + oc(v^*) - vF(v^*)^{N-1}$; and since we are concerned only with *expected* budget properties, by Lemma 4 it is sufficient to restrict attention to the Groves(X) auction that pays each bidder fixed participation payment X. By Proposition 4, if v^* is a cutoff equilibrium then $X \leq c(0) + oc(v^*)$. Also, by the derivation in the proof of Theorem 9,

$$X = c(v^*) + oc(v^*) - vF(v^*)^{N-1} = oc'(v^*) \frac{1 - F(v^*)}{f(v^*)} \tag{68}$$

Putting these facts together, there exists an allocatively-efficient auction that achieves symmetric Bayes-Nash cutoff equilibrium v^* without running a deficit in expectation if and only if $X \leq c(0)+oc(v^*)$, i.e., $c(v^*)-c(0) \leq v^* F(v^*)^{N-1}$, and the expected sum of the participation payments made by Groves(X) does not exceed the expected Vickrey auction revenue in cutoff equilibrium v^*, i.e.:

$$N(1 - F(v^*))X \leq \int_{v^*}^{1} h_N^2(x) x \, dx \tag{69}$$

Substituting for X based on Eq. (68) and rearranging, this is:

$$oc'(v^*) \leq \frac{1}{N(1 - F(v^*))} \frac{f(v^*)}{1 - F(v^*)} \int_{v^*}^{1} h_N^2(x) x \, dx \tag{70}$$

Alternatively if the efficient cutoff level is 0 (full-participation), the minimal participation payment required to achieve it equals $c(0) + oc(0)$ if c is decreasing, and $c(1) + oc(0)$ if c is increasing. This payment will be made to all N agents (since all participate in equilibrium), and ex ante no-deficit is satisfied if and only if the sum of such payments is no greater than the expected revenue of the Vickrey auction, i.e.:

$$N\big(oc(0) + \max\{c(0), c(1)\}\big) \leq \int_0^1 h_N^2(x)x\,dx \tag{71}$$

This completes the proof. □

Note that if no ex ante no-deficit and allocatively-efficient mechanism exists, then no ex post no-deficit and allocatively-efficient mechanism exists. The bound of Theorem 11 is loose, and in some cases the ex ante bound will bind tighter than the bound of Theorem 11; therefore we can amend Theorem 11 to say that no ex post no-deficit and allocatively-efficient mechanism achieving interior equilibrium cutoff v^* exists if:

$$oc'(v^*) > \min\left\{\frac{f(v^*)}{N(1-F(v^*))^2}\int_{v^*}^1 h_N^2(x)x\,dx,\ \frac{f(v^*)}{1-F(v^*)}\int_{v^*}^1 h_{N-1}^2(x)x\,dx\right\} \tag{72}$$

Figure 6 illustrates how the analysis in the proofs of Theorems 11 and 12 can be applied to example 1. Lower-bounds on the cutoff levels achievable in equilibrium under the two no-deficit constraints are depicted. Bearing in mind that the example posits uniformly distributed values, for all $v \in [0,1]$, $c(v) - c(0) = 0 \leq vF(v)^{N-1} = v^N$, and so an exact characterization of the interior cutoff levels that can be achieved under the ex ante constraint is the $v \in (0,1)$ such that:

$$c(v) + oc(v) - v^N \leq \frac{1}{N(1-v)}\int_v^1 h_N^2(x)x\,dx \tag{73}$$

And an exclusionary criterion under the ex post no-deficit constraint is the following:

$$c(v) + oc(v) - v^N > \min\left\{\frac{1}{N(1-v)}\int_v^1 h_N^2(x)x\,dx,\ \int_v^1 h_{N-1}^2(x)x\,dx\right\} \tag{74}$$

In both cases these constraints provide a dividing line: for some v_{ea}, no $v \in [0, v_{ea})$ is achievable under the ex ante no-deficit constraint and all $v \in [v_{ea}, 1]$ are; for some v_{ep}, no $v \in [0, v_{ep})$ is achievable under the ex post no-deficit constraint—since the ex post bound is not tight, the theorem is silent regarding which cutoff levels greater than or equal to v_{ep} are achievable. In the example we see that the efficient cutoff level is unachievable under the ex post no-deficit constraint, but *is* achievable if we weaken the no-deficit constraint to ex ante; the Groves(0.061) auction achieves the efficient cutoff level in a unique symmetric Bayes-Nash cutoff equilibrium and is ex ante no-deficit.

It is interesting to note that if we go outside of the class of allocatively-efficient auctions and apply a positive reserve price (see Theorem 4, which can trivially be

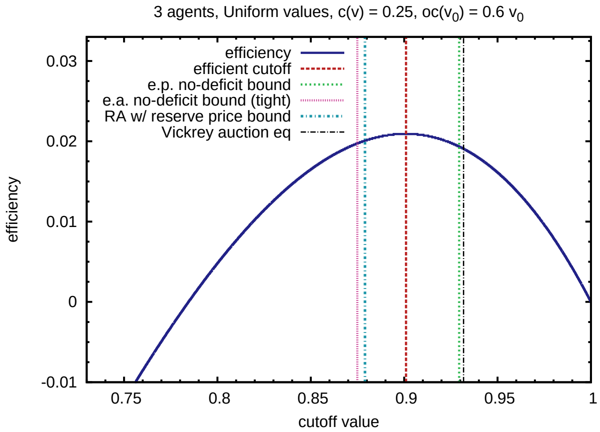

Figure 6: Efficiency as a function of symmetric cutoff level, with the following cutoff levels annotated with vertical lines (from left to right): the minimum cutoff achievable by an ex ante no-deficit allocatively-efficient auction, the minimum achievable by RA_1 with a reserve price, the optimal cutoff, the minimum achievable by an ex post no-deficit allocatively-efficient auction, and the cutoff achieved in the Vickrey auction's unique symmetric equilibrium. Efficiency is impossible to achieve here with an allocatively-efficient ex post no-deficit auction; it is achievable with an ex ante no-deficit auction or an ex post no-deficit auction that applies a reserve price.

extended to the obscurity costs setting), then the efficient cutoff equilibrium can be achieved here with an auction that never runs a deficit in equilibrium, namely, the RA_1 auction with reserve price 0.632. The cutoff level achieved in the unique symmetric Bayes-Nash cutoff equilibrium of this auction is 0.9, corresponding exactly to the red dashed efficiency line in Figure 6. A bound on the cutoff levels achievable in equilibrium in an RA_ρ auction with reserve price is illustrated—it is a bound in the sense that *all* cutoff levels between this level and the Vickrey auction equilibrium level *are* achievable;[18] cutoff levels outside of this range may not be.

While the RA_1 auction with reserve price 0.632 does not fall within our definition of *allocatively-efficient* because it may choose an inefficient outcome (specifically, not allocating the good at all if the high bid fails to meet the reserve price), the efficient outcome always occurs in equilibrium: in the first stage of the game when participation decisions are made, no agent with a value below the reserve price will choose to participate;[19] then in the dominant strategy outcome of the second stage all bids will exceed the reserve price, and thus non-allocation will never result. This auction may fall short of the no-reserve-price auction only in *off-equilibrium* play: if only agents with value below the equilibrium cutoff level choose to participate in

[18]For each cutoff v in this range, there exists an $r \in [0, v]$ and a $\rho \in [0, 1]$ such that RA_ρ with reserve price r has symmetric cutoff equilibrium v.

[19]This does not hold for all possible reserve price choices, but does for those ≤ 0.632.

the first stage, then in the second stage an inefficient outcome (with respect to the options available given the participation decisions) will result.

It is not the case that there always exists an auction that is efficient and ex ante no-deficit. Consider:

> **Example 2:** There are 3 agents and values are uniformly distributed over $[0, 1]$. There is a fixed entry cost $c(v) = 0.2$ for all values $v \in [0, 1]$, and there are linear obscurity costs $oc(v_0) = 0.6v_0$, $\forall v_0 \in [0, 1]$.

Example 2 is identical to example 1 except that the fixed entry cost has been decreased somewhat, from 0.25 to 0.2. In this case the efficient cutoff level is full-participation. To achieve full-participation and efficient allocation of the good in equilibrium, the auction must provide a participation payment of at least $\max_{v\in[0,1]}[c(v) + oc(0)] = 0.2$ (otherwise an agent with value very close to 0 would be better off not participating). Then, in equilibrium the expected revenue of the auction that makes this minimal participation payment equals:

$$\int_0^1 h_N^2(x)x\,dx - N \cdot 0.2 = \frac{N-1}{N+1} - N \cdot 0.2 \tag{75}$$

$$= \frac{2}{4} - 0.6 < 0 \tag{76}$$

A deficit results in expectation.

5 Beyond linear entry cost functions

The redistribution payments made by RA_ρ impose a need for the linear structure of cost functions in proving the equilibrium results of the previous sections. But in the case of the Vickrey auction, we can go well beyond linear costs and still demonstrate existence of a unique symmetric cutoff equilibrium.

Definition 6. *(*REGULAR COST FUNCTION WITH NEGATIVE PROPORTIONAL GRADIENT*) A cost function $c : [0, 1] \to [0, 1]$ is regular if and only if $\forall v \in [0, 1]$, $c(v) \in (0, 1)$; and c has negative proportional gradient if and only if, $\forall v \in (0, 1]$, c is differentiable at v and:*

$$\left(\frac{c(v)}{v}\right)' \leq 0 \tag{77}$$

Theorem 13. *Consider arbitrary regular entry cost function c with negative proportional gradient, and values with arbitrary continuous density function with support $[0, 1]$. $\forall v_0 \in (0, 1)$, v_0 is a symmetric Bayes-Nash cutoff equilibrium of the Vickrey auction if and only if $v_0 G(v_0) = c(v_0)$. If c is concave there is a unique symmetric Bayes-Nash cutoff equilibrium and it is efficient.*

Proof. See Appendix. □

Negative proportional gradient is in fact close to (but not quite) a *necessary* condition for there to be a symmetric cutoff equilibrium: as the proof of Theorem 13 exposes, for any cutoff level v_0, if there exists a $\underline{v} < v_0$ such that $\frac{c(\underline{v})}{\underline{v}} < \frac{c(v_0)}{v_0}$, then v_0 cannot be a cutoff equilibrium.[20] A wide array of cost functions have negative proportional gradient. Important subclasses are all decreasing functions and all linear functions (decreasing or increasing).

Proposition 5. *All regular monotonically decreasing cost functions and all regular linear cost functions have negative proportional gradient.*

Proof. Taking the derivative of $\frac{c(v)}{v}$, we can see that negative proportional gradient is equivalent to the condition that $\forall v \in (0,1]$, $\frac{vc'(v)-c(v)}{v^2} \leq 0$, which holds if and only if:

$$vc'(v) \leq c(v) \tag{78}$$

For any decreasing function c, c' is non-positive; then since c has range $(0,1)$, the inequality follows. If c is a regular linear function, then there exists some $b, k \in \Re$ such that $c(v) = b + kv$, $\forall v \in [0,1]$. Plugging this into Eq. (78), we have:

$$v \cdot (b + kv)' \leq b + kv, \text{ i.e.,} \tag{79}$$

$$kv \leq b + kv \tag{80}$$

This is guaranteed to hold by regularity of c. □

So, for instance, an agent's cost of entry might rise proportionally with his value, e.g., $c(v) = 0.1 + \frac{v}{2}$, or decrease sharply, e.g., $c(v) = 0.9 - v^2$. A unique symmetric cutoff equilibrium will obtain for all such cost functions under the Vickrey auction.

6 Conclusion

In most real-world auction settings, individuals are not captives, simply trying to do their best in a situation they have no choice but to engage with. More often if they participate it is by free choice, made because the auction provides expected utility that is greater than what they could obtain by spending their time and efforts elsewhere. Modeling the foregoing of this outside option as a *cost* (and there may be other more direct costs associated with participation as well), we find that the form of the costs is central to predicting the circumstances under which agents will participate and the efficiency level that will result.

In this paper we started by extending previously known results, showing that for costs that are an arbitrary linear function of value (rather than simply fixed), a broad space of auctions has a unique symmetric equilibrium where each agent will participate if and only if his value is above some threshold. Within this space, the Vickrey auction is superlative when costs are solely a function of value and do not

[20]It is not a necessary condition since negative proportional gradient goes further to require that $\frac{c(\underline{v})}{\underline{v}} \geq \frac{c(\overline{v})}{\overline{v}}$ for *every* $\overline{v} \in (\underline{v}, 1]$.

depend on the participation rate. But a natural extended cost model takes into account the fact that participating in "obscure" auctions often brings additional costs: costs to find the auction, to prepare a bid, to engage in a system lacking the social seal of approval that mass participation brings, etc. In this case the Vickrey auction is inefficient because it yields under-participation; we demonstrated that redistribution auctions improve efficiency without ever running a deficit. We went on to consider optimally efficient auction design in such positive participation externality settings. We characterized the symmetric equilibria here, derived an algorithm for identifying a *welfare-optimal* auction, and demonstrated that perfect efficiency may or may not be achievable in the context of an auction that does not run a deficit. Limitations of our analysis include the restriction to allocatively-efficient auctions, and the emphasis on unique symmetric equilibria when asymmetric equilibria may also be present.

While we formally considered only a one-shot setting, perhaps some broader lessons for market design can be drawn. Our results can be interpreted as suggestive that for "upstart" markets it may be advisable to choose a mechanism that channels much of the social surplus to buyers (e.g., a redistribution auction); as the market matures and obscurity costs decrease, the impact on efficiency of switching to a higher-revenue mechanism will be less dramatic. In this paper we were concerned primarily with efficiency while meeting a hard budget-balance constraint, but future work may consider the precise revenue impact of using redistribution to spur extra participation. Although redistribution very explicitly subtracts revenue from the seller, the extra competition that increased participation brings provides a countervailing factor that increases revenue, which may significantly mitigate—or in very special cases, even eliminate—overall revenue loss.

Even when the decrease in revenue from moving to an efficient redistribution auction is severe, there are important settings where this move is appropriate and plausible, for instance allocation of a public resource by the government. Consider the problem of efficiently allocating a newly available band of wireless spectrum among, say, a group of cellular network operators. Researching and understanding the nature of the auction, preparing an estimate of value, going through the logistics of bidding, etc., are all costly demands of participation in the auction. These logistical costs may decrease as more companies decide to participate—for instance, a third-party bid consultancy may be founded to help companies estimate value—in which case, we showed, the efficient allocation mechanism will involve participation bonuses for bidders. Efficient allocation is the government mandate here, rather than revenue generation, and as long as there is no deficit there is no obstacle to redistributing Vickrey auction revenue in the service of social welfare. This paper provided an analysis of how best to do so.

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Appendix

Theorem 4. *Consider arbitrary regular linear entry cost function c and values with arbitrary continuous density function with support $[0,1]$. Let v_0 be the unique symmetric Bayes-Nash cutoff equilibrium of RA_ρ with no reserve price (where ρ is restricted to choices ensuring that such an equilibrium exists and is unique). There exists an $r > 0$ such that RA_ρ with reserve price r achieves greater expected allocation value in a unique symmetric Bayes-Nash cutoff equilibrium, with no deficit, if $F(v_0) < \frac{N-1}{N-3+\frac{2}{\rho}}$.*

Proof. Consider arbitrary regular linear entry cost function c and values with arbitrary continuous density function with support $[0,1]$. For arbitrary $v \in [0,1]$ and $r \geq 0$, let $\pi(v,r) = R_\rho(v,0) + (v-r)G(v) - c(v)$. Consider arbitrary $\rho \in (0,1]$ such that $\pi(v_0, 0) = 0$ has a unique solution $v_0 \in (0,1)$ with $c(0) + oc(v_0) \geq R_\rho(v_0, 0)$. The proof of Theorem 7 will demonstrate that such a ρ exists, and that RA_ρ with no reserve price then has a unique symmetric Bayes-Nash cutoff equilibrium $v_0 \in (0,1)$.

Adding reserve price r, an $r > 0$ can always be chosen sufficiently small such that the RA_ρ auction with reserve price r has a unique symmetric cutoff equilibrium $v' \in (r, 1)$. This follows from the fact that as r goes to 0, the values of $\pi(v,r)$ and $\pi(v,0)$ at every $v \in [0,1]$ converge, which implies that for small enough r, $\pi(v', r) = 0$ at exactly one point $v' \in (0,1)$, since $\pi(v,0) = 0$ at exactly one $v \in (0,1)$. Moreover, r can be chosen sufficiently small such that the v' solving $\pi(v', r) = 0$ satisfies $v' \geq r$; this holds because as r goes to 0, v' converges to v, and $v > 0$.[21] In cutoff equilibrium v', RA_ρ with reserve price r will never yield a deficit (regardless of the agents' values) by Proposition 2.

Now, for an $r > 0$ that yields a unique symmetric cutoff equilibrium $v' \in [r, 1)$, the cutoff level v' will be strictly less than v_0 (i.e., $v' \in [r, v_0)$) if $\pi(v_0, r) > \pi(v_0, 0)$, as this implies that at v_0, $\pi(\cdot, r)$ has already switched from negative to positive while $\pi(\cdot, 0)$ is just on the verge of doing so (recall that $\pi(v,r)$ and $\pi(v,0)$ are both negative at $v = 0$, positive at $v = 1$, and equal to 0 only at one point each). Therefore to finish proving the theorem it is sufficient to show that $\pi(v_0, r) > \pi(v_0, 0)$, $\forall r \in (0, v_0)$.

[21] In practice there will often be a large range of choices of r—rather than just those extremely close to 0—that preserve uniqueness of a symmetric equilibrium v' (and also satisfy $v' \geq r$); but this argument shows that *some* positive r that does so always exists.

We see that for arbitrary $r \in \Re^+$, $\pi(v_0, r) - \pi(v_0, 0) =$

$$[R_\rho(v_0, r) + (v_0 - r)G(v_0) - c(v_0)] - [R_\rho(v_0, 0) + v_0 G(v_0) - c(v_0)] \tag{81}$$
$$= R_\rho(v_0, r) - R_\rho(v_0, 0) - rG(v_0) \tag{82}$$
$$= \mathcal{P}(N-1, 0, v_0)\rho r + \mathcal{P}(N-1, 1, v_0)\frac{\rho}{2} r \tag{83}$$
$$+ \sum_{m=2}^{N-1} \mathcal{P}(N-1, m, v_0)\frac{\rho}{m+1}\int_{v_0}^{1} h_m^2(x, v_0) x \, dx \tag{84}$$
$$- \sum_{m=2}^{N-1} \mathcal{P}(N-1, m, v_0)\frac{\rho}{m+1}\int_{v_0}^{1} h_m^2(x, v_0) x \, dx \; - \; rF(v_0)^{N-1} \tag{85}$$
$$= r\Big(\mathcal{P}(N-1, 0, v_0)\rho + \mathcal{P}(N-1, 1, v_0)\frac{\rho}{2} - F(v_0)^{N-1}\Big) \tag{86}$$

Thus $\pi(v_0, r) > \pi(v_0, 0)$ if and only if $\mathcal{P}(N-1, 0, v_0)\rho + \mathcal{P}(N-1, 1, v_0)\frac{\rho}{2} - F(v_0)^{N-1} > 0$. We have:

$$\mathcal{P}(N-1, 0, v_0)\rho + \mathcal{P}(N-1, 1, v_0)\frac{\rho}{2} - F(v_0)^{N-1} \tag{87}$$
$$= \binom{N-1}{0}(1 - F(v_0))^0 F(v_0)^{N-1}\rho + \binom{N-1}{1}(1 - F(v_0))^1 F(v_0)^{N-2}\frac{\rho}{2} - F(v_0)^{N-1} \tag{88}$$
$$= F(v_0)^{N-1}\rho + (N-1)(1 - F(v_0))F(v_0)^{N-2}\frac{\rho}{2} - F(v_0)^{N-1} \tag{89}$$
$$= \frac{\rho F(v_0)^{N-1}}{2}\Big[2 + (N-1)(1 - F(v_0))F(v_0)^{-1} - \frac{2}{\rho}\Big] \tag{90}$$
$$= \frac{(N-1)\rho F(v_0)^{N-1}}{2}\Big[\frac{2(1 - 1/\rho)}{N-1} - 1 + \frac{1}{F(v_0)}\Big] \tag{91}$$
$$= \frac{(N-1)\rho F(v_0)^{N-2}}{2}\Big[F(v_0)\Big(\frac{2(1 - 1/\rho)}{N-1} - 1\Big) + 1\Big] \tag{92}$$
$$= \frac{(N-1)\rho F(v_0)^{N-2}}{2}\Big(\frac{2(1 - 1/\rho)}{N-1} - 1\Big)\Big[F(v_0) + \frac{1}{\frac{2(1-1/\rho)}{N-1} - 1}\Big] \tag{93}$$
$$= \frac{(N-1)\rho F(v_0)^{N-2}}{2}\Big(\frac{2(1 - 1/\rho)}{N-1} - 1\Big)\Big[F(v_0) + \frac{N-1}{2(1 - 1/\rho) - N + 1}\Big] \tag{94}$$
$$= \frac{(N-1)\rho F(v_0)^{N-2}}{2}\Big(\frac{2(1 - 1/\rho)}{N-1} - 1\Big)\Big[F(v_0) - \frac{N-1}{N - 3 + 2/\rho}\Big] \tag{95}$$
$$\tag{96}$$

The first term is positive and the second term is negative, so the formula as a whole will be greater than 0 if and only if the third term is negative, i.e., if and only if:

$$F(v_0) < \frac{N-1}{N - 3 + 2/\rho} \tag{97}$$

This completes the proof. □

Theorem 6. *Consider arbitrary regular linear entry cost function c, arbitrary obscurity cost function oc, values with arbitrary continuous density function with support $[0,1]$, and arbitrary $r \geq 0$. $\forall v_0 \in (r,1)$, v_0 is a symmetric Bayes-Nash cutoff equilibrium of RA_ρ with reserve price r if and only if $R_\rho(v_0,r) + (v_0 - r)G(v_0) = c(v_0) + oc(v_0)$ and $c(0) + oc(v_0) \geq R_\rho(v_0,r)$.*

Proof. Consider arbitrary linear cost function $c(v) = b + kv$, arbitrary obscurity cost function oc, values with arbitrary continuous density function with support $[0,1]$, and arbitrary $r \geq 0$. For arbitrary $v_0 \in (r,1)$, note that if $R_\rho(v_0,r)+(v_0-r)G(v_0) \neq c(v_0) + oc(v_0)$, then v_0 could not be a cutoff equilibrium (by a trivial extension of Lemma 1 to the obscurity costs setting); and if $c(0)+oc(v_0) < R_\rho(v_0,r)$, v_0 could not be a cutoff equilibrium because an agent with value 0 would obtain positive utility from participation. Thus, for arbitrary $v_0 \in (r,1)$, $R_\rho(v_0,r)+(v_0-r)G(v_0) = c(v_0)+oc(v_0)$ and $c(0) + oc(v_0) \geq R_\rho(v_0,r)$ necessarily hold if v_0 is a cutoff equilibrium.

Now assume a $v_0 \in (r,1)$ satisfying $R_\rho(v_0,r) + (v_0 - r)G(v_0) = c(v_0) + oc(v_0)$ and $c(0) + oc(v_0) \geq R_\rho(v_0,r)$ exists, and assume all agents other than some i play the posited equilibrium strategy. If i participates, truthfulness will be a dominant strategy, and so the theorem holds if i obtains non-negative net utility if and only if his value is at least v_0. If i's value equals v_0, if he participates his expected utility is:

$$u_i(v_0) = R_\rho(v_0,r) + (v_0 - r)G(v_0) - c(v_0) - oc(v_0) = 0 \tag{98}$$

Bear in mind that $G(v_0) = \frac{c(v_0)+oc(v_0)-R_\rho(v_0,r)}{v_0-r}$, and that this implies $c(v_0) + oc(v_0) - R_\rho(v_0,r) > 0$ since $G(v_0) > 0$ and $v_0 - r > 0$. If i's value is $\underline{v} < v_0$, if he participates his expected utility is:

$$\begin{aligned}
u_i(\underline{v}) &= R_\rho(v_0,r) + (\underline{v} - r)G(v_0) - c(\underline{v}) - oc(v_0) && (99)\\
&= R_\rho(v_0,r) + \frac{\underline{v} - r}{v_0 - r}(c(v_0) + oc(v_0) - R_\rho(v_0,r)) - c(\underline{v}) - oc(v_0) && (100)\\
&= \frac{\underline{v} - r}{v_0 - r}(b + kv_0 + oc(v_0) - R_\rho(v_0,r)) - (b + k\underline{v} + oc(v_0) - R_\rho(v_0,r)) && (101)\\
&\leq \frac{\underline{v}}{v_0}(b + kv_0 + oc(v_0) - R_\rho(v_0,r)) - (b + k\underline{v} + oc(v_0) - R_\rho(v_0,r)) && (102)\\
&= \frac{\underline{v}}{v_0}(b + oc(v_0) - R_\rho(v_0,r)) - (b + oc(v_0) - R_\rho(v_0,r)) \leq 0 && (103)
\end{aligned}$$

We use the fact that $v_0 - r > 0$ and $b + oc(v_0) = c(0) + oc(v_0) \geq R_\rho(v_0,r)$. So i cannot gain by participating for any $\underline{v} < v_0$.

Now consider the case where i's value is $\overline{v} > v_0$. Let $p(y)$ equal y if $y \geq v_0$ and

r otherwise. If i participates then his expected utility is:

$$u_i(\overline{v}) = R_\rho(v_0, r) + \int_0^{\overline{v}} [\overline{v} - p(y)]g(y)\,dy - c(\overline{v}) - oc(v_0) \tag{104}$$

$$= R_\rho(v_0, r) + \int_0^{v_0} [\overline{v} - p(y)]g(y)\,dy - c(\overline{v}) - oc(v_0) + \int_{v_0}^{\overline{v}} [\overline{v} - p(y)]g(y)\,dy \tag{105}$$

$$= R_\rho(v_0, r) + (\overline{v} - r)G(v_0) - c(\overline{v}) - oc(v_0) + \int_{v_0}^{\overline{v}} [\overline{v} - p(y)]g(y)\,dy \tag{106}$$

$$\geq R_\rho(v_0, r) + (\overline{v} - r)G(v_0) - c(\overline{v}) - oc(v_0) \tag{107}$$

$$= R_\rho(v_0, r) + \frac{\overline{v} - r}{v_0 - r}(c(v_0) + oc(v_0) - R_\rho(v_0, r)) - c(\overline{v}) - oc(v_0) \tag{108}$$

$$= \frac{\overline{v} - r}{v_0 - r}(b + kv_0 + oc(v_0) - R_\rho(v_0, r)) - (b + k\overline{v} + oc(v_0) - R_\rho(v_0, r)) \tag{109}$$

$$\geq \frac{\overline{v}}{v_0}(b + kv_0 + oc(v_0) - R_\rho(v_0, r)) - (b + k\overline{v} + oc(v_0) - R_\rho(v_0, r)) \tag{110}$$

$$= \frac{\overline{v}}{v_0}(b + oc(v_0) - R_\rho(v_0, r)) - (b + oc(v_0) - R_\rho(v_0, r)) + \geq 0 \tag{111}$$

Again the final inequality makes use of the fact that $b + oc(v_0) = c(0) + oc(v_0) \geq R_\rho(v_0, r)$. The move from Eq. (106) to Eq. (107) holds because $p(y) < \overline{v}$ for all $y < \overline{v}$, and Eq. (107) to Eq. (108) holds by the fact that v_0 is defined such that $R_\rho(v_0, r) + (v_0 - r)G(v_0) = c(v_0) + oc(v_0)$. The final two moves hold since $v_0 - r > 0$ and $\overline{v} > v_0$. So i cannot gain from *not* participating when his value is greater than v_0, which completes the proof. □

Lemma 3. *For arbitrary continuous cost function c and obscurity cost function oc, and values with arbitrary continuous density function with support $[0, 1]$, if $v_0 \in (0, 1)$ is a symmetric Bayes-Nash cutoff equilibrium of a Groves auction, in that equilibrium the expected participation payment received by each bidder equals $c(v_0) + oc(v_0) - v_0G(v_0)$.*

Proof. Consider arbitrary continuous cost function c, obscurity cost function oc, and values with arbitrary continuous density function with support $[0, 1]$. Consider arbitrary Groves auction with symmetric Bayes-Nash cutoff equilibrium $v_0 \in (0, 1)$, and $\forall i \in I$, let X_i denote the expected participation payment received by agent i if he participates and all other agents play the equilibrium strategy (X_i is independent of i's value and bid by the definition of Groves auctions). Assume the lemma fails. Then for some $i \in I$, $X_i + v_0G(v_0) - c(v_0) - oc(v_0) \neq 0$. If $X_i + v_0G(v_0) - c(v_0) - oc(v_0) < 0$, then if i has value v_0, since in equilibrium he will win the auction only if no other agent participates, his expected utility equals:

$$X_i + v_0G(v_0) - c(v_0) - oc(v_0) < 0, \tag{112}$$

and so setting a higher cutoff-level is a beneficial deviation, contradicting v_0 as a symmetric cutoff equilibrium.

Alternatively if $X_i + v_0 G(v_0) - c(v_0) - oc(v_0) = \gamma > 0$, then consider i's expected utility from participation if he has value $v_0 - \epsilon$, for some small $\epsilon > 0$. Noting that here too in equilibrium i will win the auction if and only if no other agent participates, the difference between the equilibrium expected utility for i with value v_0 and value $v_0 - \epsilon$ is:

$$[X_i + v_0 G(v_0) - c(v_0) - oc(v_0)] - \tag{113}$$

$$[X_i + (v_0 - \epsilon) G(v_0) - c(v_0 - \epsilon) - oc(v_0)] \tag{114}$$

$$= \epsilon G(v_0) - [c(v_0) - c(v_0 - \epsilon)] \tag{115}$$

Because cost function c is continuous, $\epsilon > 0$ can be chosen small enough such that this difference (Eq. (115)) is strictly less than γ. This implies that the expected utility of participating with value $v_0 - \epsilon$ is greater than 0, contradicting v_0 as a symmetric cutoff equilibrium since when i has value $v_0 - \epsilon$ he would benefit from participating. □

Proposition 4. *Consider arbitrary regular linear entry cost function c and obscurity cost function oc such that $c(v) + oc(v) \in (0,1)$, $\forall v \in [0,1]$, values with arbitrary continuous density function with support $[0,1]$, and arbitrary $X \in \Re$. The following is an exhaustive characterization of the symmetric Bayes-Nash cutoff equilibria of the Groves(X) auction:*

- *$\forall v_0 \in (0,1)$, if $X = c(v_0) + oc(v_0) - v_0 G(v_0)$ and $X \leq c(0) + oc(v_0)$, then v_0 is a symmetric Bayes-Nash cutoff equilibrium.*
- *If $X \geq c(v) + oc(0) - (N-1)\int_0^v F(x)^{N-2} f(x)(v-x)\,dx$, $\forall v \in [0,1]$, full-participation is a symmetric Bayes-Nash cutoff equilibrium.*

If oc is concave and non-decreasing and $vG(v)$ is convex over $[0,1]$: if $0 \leq X < c(0) + oc(0)$, there is a unique symmetric cutoff equilibrium and it is interior (i.e., $\in (0,1)$); if $X \geq c(0) + oc(0)$, there may be between 0 and 3 symmetric cutoff equilibria, with a maximum of two that are interior.

Proof. Consider the set V of $v_0 \in (0,1)$ such that $c(v_0) + oc(v_0) - v_0 G(v_0) = X$. Lemma 3 eliminates all other values in $(0,1)$ as candidates for a symmetric cutoff equilibrium. For each $v_0 \in V$, if $X \leq c(0) + oc(v_0)$ then the proof of Theorem 6 (replacing $R_\rho(v_0, r)$ with X) applies to show that v_0 is a symmetric cutoff equilibrium. If the obscurity cost function is concave and $vG(v)$ is convex, then $c(v) + oc(v) - vG(v)$ is concave and therefore $|V| \leq 2$; furthermore if $X \in [0, c(0) + oc(0))$ then $|V| = 1$, so letting v_0 denote the sole element of V, if oc is non-decreasing then $X < c(0) + oc(0)$ implies $X < c(0) + oc(v_0)$ and thus v_0 is a cutoff equilibrium.

If $X \geq c(v) + oc(0) - (N-1)\int_0^v F(x)^{N-2} f(x)(v-x)\,dx$, $\forall v \in [0,1]$ (note that this implies $X \geq c(0) + oc(0)$), then full-participation will also be an equilibrium since every agent's participation payment plus expected Vickrey-auction surplus, regardless of his value, will exceed his entry cost; otherwise it will not be.

Finally, note that under no circumstances will full non-participation (cutoff level 1) be a symmetric equilibrium since for arbitrary $X \geq 0$, there exists a $v \in (0, 1)$ great enough such that $X + v - c(v) - oc(1) > 0$; if all other agents are non-participating, an agent with value v would gain from participation. $\square$

Theorem 10. *Consider values uniformly distributed over* $[0, 1]$*, and arbitrary regular linear entry cost function* c *and linear obscurity cost function* oc *such that* $c(v) + oc(v') \in (0, 1)$, $\forall v, v' \in [0, 1]$. *If* $\exists v \in (0, 1)$ *such that* $c(v) + oc(v) - v^N = oc'(v)(1 - v)$*, let* $v_0 = \max\{v \in (0, 1) \,|\, c(v) + oc(v) - v^N = oc'(v)(1 - v)\}$ *and* $v^* = \min\{v \in [v_0, 1) \,|\, c(v) - c(0) \leq v^N\}$*; among all allocatively-efficient auctions, letting* $X = c(v^*) + oc(v^*) - v^{*N}$*, the Groves*$(X)$ *auction is welfare-optimal if:*

$$oc(0) - (1 - v^*)oc(v^*) > v^* c(v^*/2) + \frac{v^{*N+1}}{N + 1} \tag{116}$$

Otherwise, or if $\nexists v \in (0, 1)$ *such that* $c(v)+oc(v)-v^N = oc'(v)(1-v)$*, the Groves*$(Y)$ *auction is welfare-optimal where* $Y = \max_{v \in [0,1]}[c(v) + oc(0)]$.

Proof. From the analysis in the proof of Theorem 9 we know that the derivative, with respect to symmetric cutoff level v_0, of the expected social welfare is $\frac{\partial}{\partial v_0} SW(v_0) =$

$$N f(v_0)\Big(c(v_0) + oc(v_0) - v_0 F(v_0)^{N-1} - oc'(v_0)\frac{1 - F(v_0)}{f(v_0)}\Big) \tag{117}$$

For F uniform over $[0, 1]$, cost function $c(v) = b + kv$, and linear obscurity cost function $oc(v) = b_2 + k_2 v$, this reduces to:

$$N\big(b + kv_0 + b_2 + k_2 v_0 - v_0^N - k_2(1 - v_0)\big) \tag{118}$$

Noting that the derivative of this, with respect to v_0, equals $N(k + 2k_2 - N v_0^{N-1})$ and is therefore decreasing, we see that Eq. (118) is concave. Since Eq. (118) is < 0 at $v_0 = 1$, this in turn implies that the social welfare function is optimized either at $v_0 = 0$ (as is clearly the case when Eq. (118) is negative across $[0, 1]$) or, assuming $\exists v \in (0, 1)$ such that $c(v) + oc(v) - v^N = oc'(v)(1 - v)$, at the *greatest* zero of Eq. (118), i.e., at:

$$v_0 = \max\{v \in (0, 1) \,|\, c(v) + oc(v) - v^N = oc'(v)(1 - v)\} \tag{119}$$

Assume $\exists v \in (0, 1)$ such that $c(v) + oc(v) - v^N = oc'(v)(1 - v)$. The v_0 of Eq. (119) may or may not be achievable as a symmetric cutoff equilibrium. By Proposition 4 we know that any $v \in (0, 1)$ will be achievable as such if and only if $c(v) + oc(v) - v^N \leq c(0) + oc(v)$, i.e., $c(v) - c(0) \leq v^N$. Since $c(v) - c(0)$ is a line and v^N is strictly convex, they either do not intersect on $(0, 1)$ (if $k \leq 0$), in which case all $v \in (0,1)$ are achievable, or they intersect at exactly one point $v' \in (0,1)$, in which case all $v \geq v'$ will be achievable while all $v < v'$ will not be. Since $SW(v)$ is decreasing for all $v > v_0$, the social welfare maximizing achievable interior symmetric cutoff equilibrium level v^* is the smallest $v \geq v_0$ such that $c(v) - c(0) \leq v^N$, i.e.,

$$v^* = \min\{v \in [v_0, 1) \,|\, c(v) - c(0) \leq v^N\} \tag{120}$$

It is possible that full-participation will be more efficient than cutoff level v^*, and thus any interior symmetric cutoff equilibrium; and full-participation can also be achieved as a symmetric equilibrium via a large enough participation payment. For any symmetric cutoff level v (including $v = 0$), expected social welfare equals $\int_v^1 NF(x)^{N-1} f(x) x\, dx - N \int_v^1 f(x) c(x)\, dx - N(1 - F(v)) oc(v)$. For uniform values and linear obscurity costs this reduces to:

$$N \int_v^1 x^N\, dx - N \int_v^1 c(x)\, dx - N(1-v)oc(v) \tag{121}$$

$$= N\left[\frac{1 - v^{N+1}}{N+1} - c(1/2) - vc(v/2) - (1-v)oc(v)\right] \tag{122}$$

Thus the difference between expected social welfare in symmetric cutoff level v^* and full-participation is:

$$N\left[\frac{1 - v^{*N+1}}{N+1} - c(1/2) - v^* c(v^*/2) - (1-v^*)oc(v^*)\right] - \tag{123}$$

$$N\left[\frac{1}{N+1} - c(1/2) - oc(0)\right] \tag{124}$$

$$= N\left[oc(0) - (1-v^*)oc(v^*) - v^* c(v^*/2) - \frac{v^{*N+1}}{N+1}\right] \tag{125}$$

Cutoff level v^* is therefore more efficient than full-participation if and only if:

$$oc(0) - (1-v^*)oc(v^*) > v^* c(v^*/2) + \frac{v^{*N+1}}{N+1} \tag{126}$$

If Eq. (126) holds then the Groves(X) auction with $X = c(v^*) + oc(v^*) - v^{*N}$ is welfare-optimal among all allocatively-efficient auctions, and if Eq. (126) does not hold (or if $\nexists v \in (0,1)$ such that $c(v) + oc(v) - v^N = oc'(v)(1-v)$) then the Groves($Y$) auction with $Y = \max_{v \in [0,1]}[c(v) + oc(0)]$ is welfare-optimal. □

Theorem 13. *Consider arbitrary regular entry cost function c with negative proportional gradient, and values with arbitrary continuous density function with support* $[0,1]$. $\forall v_0 \in (0,1)$, *v_0 is a symmetric Bayes-Nash cutoff equilibrium of the Vickrey auction if and only if* $v_0 G(v_0) = c(v_0)$. *If c is concave there is a unique symmetric Bayes-Nash cutoff equilibrium and it is efficient.*

Proof. Consider arbitrary cost function c with negative proportional gradient, values with arbitrary continuous density function with support $[0,1]$, and the Vickrey auction. For arbitrary $v_0 \in (0,1)$, by Lemma 1, $v_0 G(v_0) = c(v_0)$ necessarily holds if v_0 is a cutoff equilibrium. Assume a $v_0 \in (0,1)$ satisfying $v_0 G(v_0) = c(v_0)$ exists. Decreasing proportional gradient implies that $\forall \underline{v}, \overline{v} \in (0,1)$ with $\underline{v} \leq v_0 \leq \overline{v}$,

$$\frac{c(\underline{v})}{\underline{v}} \geq \frac{c(v_0)}{v_0} \geq \frac{c(\overline{v})}{\overline{v}} \tag{127}$$

Assume all agents other than some i play the posited equilibrium strategy. If i participates, truthfulness will be a dominant strategy, and so the theorem holds if i obtains non-negative net utility if and only if his value is at least v_0. If i's value equals v_0, if he participates his expected utility is:

$$u_i(v_0) = v_0 G(v_0) - c(v_0) = 0, \tag{128}$$

If i's value is $\underline{v} < v_0$, if he participates his expected utility is:

$$\begin{aligned} u_i(\underline{v}) &= \underline{v} G(v_0) - c(\underline{v}) && (129)\\ &= \underline{v}\frac{c(v_0)}{v_0} - c(\underline{v}) \leq 0, && (130) \end{aligned}$$

where the inequality holds by Eq. (127). So i cannot gain by participating for any $\underline{v} < v_0$.

Now consider the case where i's value is $\overline{v} > v_0$. Let $p(y)$ equal y if $y \geq v_0$ and 0 otherwise. If i participates then his expected utility is:

$$\begin{aligned} u_i(\overline{v}) &= \int_0^{\overline{v}} [\overline{v} - p(y)] g(y)\, dy - c(\overline{v}) && (131)\\ &= \int_0^{v_0} [\overline{v} - p(y)] g(y)\, dy - c(\overline{v}) + \int_{v_0}^{\overline{v}} [\overline{v} - p(y)] g(y)\, dy && (132)\\ &= \overline{v} G(v_0) - c(\overline{v}) + \int_{v_0}^{\overline{v}} [\overline{v} - p(y)] g(y)\, dy && (133)\\ &\geq \overline{v} G(v_0) - c(\overline{v}) && (134)\\ &= \frac{\overline{v}}{v_0} c(v_0) - c(\overline{v}) \geq 0 && (135) \end{aligned}$$

Again the inequality holds by Eq. (127). So i cannot gain from *not* participating when his value is greater than v_0.

There always exists a v_0 such that $v_0 G(v_0) = c(v_0)$ (by regularity of c and support $[0,1]$ of f). Considering the case of concave c, there exists exactly one such v_0 by Lemma 2, which implies a unique symmetric cutoff equilibrium. Moreover, since $\frac{\partial}{\partial v}\mathbb{E}[SW(v)] = Nf(v)\big(c(v) - vG(v)\big)$, and since $c(v) > vG(v)$ for all $v \in [0, v_0)$ and $c(v) < vG(v)$ for all $v \in (v_0, 1]$ in the case of concave c, v_0 is the efficient cutoff level. □